

## FRAÏSSÉ LIMITS IN FUNCTIONAL ANALYSIS

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**ABSTRACT.** We provide a unified approach to Fraïssé limits in functional analysis, including the Gurarij space, the Poulsen simplex, and their noncommutative analogs. We obtain in this general framework many known and new results about the Gurarij space and the Poulsen simplex, and at the same time establish their noncommutative analogs. Particularly, we construct noncommutative analogs of universal operators in the sense of Rota.

## 1. INTRODUCTION

Classical Fraïssé theory studies *countable homogeneous structures*. A countable structure is homogeneous if any partial isomorphism between two finitely generated substructures extends to an automorphism of the whole structure. The foundational result of *Fraïssé theory*, obtained by Fraïssé in [43], implies that a countable homogeneous structure is completely determined by its *age*. (The age of a countable structure is the collection of all its finitely generated substructures.) The classes of finitely generated structures that arise as ages of countable homogeneous structures are now called Fraïssé classes [43]. The last fifteen years have seen a renewed interest in countable homogeneous structures and Fraïssé theory in view of the relations with *Ramsey theory* and *topological dynamics*. Indeed it was established in [61] that age of a countable homogeneous structure  $S$  satisfies the Ramsey property if and only if the automorphism group of  $S$  is *extremely amenable*. (A topological group is extremely amenable if any continuous action on a compact Hausdorff space has a fixed point.) This fact, known as Kechris-Pestov-Todorćević (KPT) correspondence, initiated a new direction of research, a survey of which can be found in [105]. One of the goals of this line of research is to prove by combinatorial methods extreme amenability of interesting Polish groups and, more generally, to compute their universal minimal compact spaces.

The main ingredient in Fraïssé's analysis is the back-and-forth method. This technique consists in building an isomorphism between two limit structures by recursively defining approximations of it on finitely generated substructures. The same basic idea is used in many arguments in functional analysis and operator algebras, where it is more often called *approximate intertwining*. This is not a coincidence. Many structures in functional analysis have been recently recognized to be of Fraïssé-theoretic nature, due to works of Ben Yaacov [7, 6], Ben Yaacov and Henson [9], Garbulińska-Węgrzyn and Kubiś [45], Kubiś [65, 66, 64], Kubiś and Kwiatkowska [67], Kubiś and Solecki [68], and unpublished work of Conley and Törnquist. This motivated Ben Yaacov [7] to generalize Fraïssé theory from the discrete to the metric setting. In this framework he established a correspondence between *metric* Fraïssé classes and separable metric structures (their *limits*) that are *approximately homogeneous*, in the sense that a partial isomorphism between finitely generated substructures can be arbitrarily well approximated by an automorphism. Other approaches to Fraïssé theory in the metric setting have been suggested in [103, 65]. Fraïssé classes arising in the theory of operator algebras have been studied in [31].

The aim of this paper is to provide a unified approach to the proof of the fundamental properties of Fraïssé limits in functional analysis, including the Gurarij space, the Poulsen simplex, and their noncommutative analogs. The Gurarij space  $\mathbb{G}$ , first constructed by Gurarij in [50], is the unique separable approximately ultrahomogeneous Banach space that is universal for separable Banach spaces [84]. The Poulsen simplex  $\mathbb{P}$ , first constructed by Poulsen in [99] is the unique nontrivial metrizable Choquet simplex with dense extreme boundary [95].

It is known since the work of Lusky [82, 85, 87, 86, 85, 83] and Lindenstrauss-Olsen-Sternfeld [79, 95] in the 1970s that the Poulsen simplex and the Gurarij space can be studied by very similar methods. This has been made precise in an unpublished work of Conley and Törnquist, who studied the Poulsen simplex by looking a

2000 *Mathematics Subject Classification*. Primary 46L07, 46A55; Secondary 46L89, 03C30, 03C98.

*Key words and phrases*. Gurarij space, Poulsen simplex, Choquet simplex, Fraïssé limit operator space, operator system, universal operator.

The author was supported by the York University Susan Mann Dissertation Scholarship, and by the ERC Starting grant no. 259527 of Goul'nara Arzhantseva. Part of this work was done while the author was visiting the Instituto de Ciencias Matemáticas in Madrid. The hospitality of the Institute is gratefully acknowledged.

the associated *function system*. A function system  $V$  is a closed subspace of a real Banach space of the form  $C(K)$  containing the function constantly equal to 1 (the *unit*). The inclusion  $V \subset C(K)$  defines on  $V$  an order structure which only depends on the norm and the unit of  $V$ . If  $K$  is any compact convex set, then the space  $A(K)$  of continuous affine functions on  $K$  is a function system. In the statement of Theorem 1.1 (5) we consider a compact convex set endowed with the norm coming from the inclusion  $K \subset A(K)^*$ .

Kadison's representation theorem [1, Theorem II.1.8] asserts that any function system  $V$  is of the form  $A(K)$ , where  $K$  is the space of unital positive linear functionals of  $V$ . Furthermore the assignment  $K \mapsto A(K)$  is a contravariant equivalence of categories from the category of compact convex sets and continuous affine maps to the category of function systems and unital positive linear maps. (Metrizable) Choquet simplices correspond to (separable) function systems that are moreover Lindenstrauss spaces. Thus for most purposes one can work with separable Lindenstrauss function systems rather than metrizable Choquet simplices. Conley and Törnquist showed that  $A(\mathbb{P})$  is the unique separable function system that is approximately homogeneous and universal for separable function systems. Thus  $A(\mathbb{P})$  has the same properties as  $\mathbb{G}$ , but in the category of function systems rather than Banach spaces. We will call  $A(\mathbb{P})$  the *Poulsen system*.

We list here some known and new facts about the Poulsen simplex  $\mathbb{P}$  that follow from our general framework and will be proved in §6.3:

**Theorem 1.1.** *Let  $\mathbb{P}$  be a nontrivial metrizable Choquet simplex with dense extreme boundary  $\partial_e \mathbb{P}$ .*

- (1) *The space  $A(\mathbb{P})$  is the unique approximately homogeneous separable function system that contains unital isometric copies of any separable function system.*
- (2)  *$\mathbb{P}$  is the unique nontrivial metrizable Choquet simplex with dense extreme boundary [79, Theorem 2.3].*
- (3) *A metrizable compact convex set is a Choquet simplex if and only if it is affinely homeomorphic to a closed proper face of  $\mathbb{P}$  [79, Theorem 2.5].*
- (4) *Any affine homeomorphism between two proper faces of  $\mathbb{P}$  extends to an affine homeomorphism of  $\mathbb{P}$  [79, Theorem 2.3].*
- (5) *the set of norm-preserving continuous affine maps from a fixed Choquet simplex  $K$  to  $\mathbb{P}$  with the property that the range is a closed proper face of  $\mathbb{P}$  is a dense  $G_\delta$  subspace of the space of continuous affine maps from  $K$  to  $\mathbb{P}$ .*
- (6) *If  $F$  is any closed proper face of  $\mathbb{P}$  affinely homeomorphic to  $\mathbb{P}$  and  $\phi : K_0 \rightarrow K_1$  is a continuous affine map between compact convex sets, then there exist continuous affine surjections  $\eta_0 : F \rightarrow K_0$  and  $\eta_1 : \mathbb{P} \rightarrow K_1$  such that  $\phi \circ \eta_0 = \eta_1|_F$ ;*
- (7) *A homeomorphism between compact subsets of  $\partial_e \mathbb{P}$  extends to an affine homeomorphism of  $\mathbb{P}$  [79, Theorem 2.5].*
- (8) *Suppose that  $F_0, F_1$  are closed proper faces of  $\mathbb{P}$ . Consider the complementary faces  $F'_0$  endowed with the compact topology induced by the functions  $a \in A(\mathbb{P})$  such that  $a|_{F_0}$  is constant, and similarly for  $F'_1$ . Then  $F'_0$  and  $F'_1$  are affinely homeomorphic [79, Theorem 2.6].*
- (9) *The canonical action of  $\text{Aut}(\mathbb{P})$  on  $\mathbb{P}$  is minimal [46, Theorem 5.2].*

One can equivalently phrase (6) by asserting that if  $F$  is any closed proper face of  $\mathbb{P}$ , then any unital positive linear map between separable function systems is a restriction-truncation to some subsystems of  $A(\mathbb{P})$  of the unital quotient mapping  $\Omega_{A(\mathbb{P})} : A(\mathbb{P}) \rightarrow A(F)$ ,  $f \mapsto f|_F$ . This can be seen as a function system version of universal operator in the sense of Rota [101].

We will prove below that one can associate to the Gurarij space  $\mathbb{G}$  a geometric object with entirely analogous properties as  $\mathbb{P}$ . By a *compact absolutely convex set* we mean a compact subset  $K$  of a locally convex topological vector space  $V$  with that is closed under absolutely convex combinations  $(x, y) \mapsto \lambda x + \mu y$  when  $|\lambda| + |\mu| \leq 1$ . A  $w^*$ -continuous function between compact absolutely convex sets is *symmetric* if it preserves the involution. One can associated to a compact absolutely convex set  $K$  the Banach space  $A_\sigma(K)$  of real-valued symmetric affine functions on  $K$ . Conversely any Banach space  $X$  arises in this way from the compact absolutely convex set  $\text{Ball}(X^*)$  [72, Lemma 1]. Here  $\text{Ball}(X^*)$  denotes the unit ball of the dual space  $X^*$  of the space  $X$ . Furthermore the assignment  $K \mapsto A_\sigma(K)$  is a contravariant equivalence of categories from the category of compact absolutely convex sets and continuous symmetric affine maps to the category of Banach spaces and linear maps of norm at most 1. The compact absolutely convex sets of the form  $\text{Ball}(X^*)$  for some Lindenstrauss space  $X$  have been characterized by Lazar in [72]; see also [33, Theorem 3.2]. We will call these compact absolutely convex sets *Lazar simplices*. The Lazar simplex  $\text{Ball}(\mathbb{G})$  corresponding to the Gurarij space will be denoted by  $\mathbb{L}$  and called the *Lusky simplex*. We will prove that  $\mathbb{L}$  plays the same role among Lazar simplices as  $\mathbb{P}$  plays among Choquet simplices. The analog of a face in this setting is the absolutely convex hull of a face, called a *biface* in [33]

and a *facial section* in [74]. In the statement of Theorem 1.2 we regard a compact absolutely convex set  $K$  as endowed with the norm coming from the inclusion  $K \subset A_\sigma(K)$ .

It will follow from our general results—see §6.1—that the analogous statements hold for the Lusky simplex when one replaces Choquet simplices with Lazar simplices and faces with bifaces:

**Theorem 1.2.** *Let  $\mathbb{L}$  be a Lazar simplex with dense extreme boundary, and set  $\mathbb{G} := A_0(\mathbb{L})$ .*

- (1)  $\mathbb{G}$  is the unique approximately homogeneous separable real Banach space that contains an isometric copy of any separable Banach space [68, 7].
- (2)  $\mathbb{L}$  is the unique Lazar simplex with dense extreme boundary [79, Theorem 6.4].
- (3) A metrizable compact absolutely convex set is a Lazar simplex if and only if it is symmetrically affinely homeomorphic to a closed proper biface of  $\mathbb{L}$  [86, Corollary 4].
- (4) Any symmetric affine homeomorphism between closed proper bifaces of  $\mathbb{L}$  extends to a symmetric affine homeomorphism of  $\mathbb{L}$ .
- (5) The set of norm-preserving continuous symmetric affine maps from a fixed Lazar simplex  $K$  to  $\mathbb{L}$  with the property that the range is a closed proper biface of  $\mathbb{L}$  is a dense  $G_\delta$  subspace of the space of continuous symmetric affine maps from  $K$  to  $\mathbb{L}$ .
- (6) If  $H$  is any closed proper biface of  $\mathbb{L}$  symmetrically affinely homeomorphic to  $\mathbb{L}$  and  $\phi : K_0 \rightarrow K_1$  is a continuous symmetric affine map between compact absolutely convex sets, then there exist symmetric continuous affine surjections  $\eta_0 : F \rightarrow K_0$  and  $\eta_1 : \mathbb{L} \rightarrow K_1$  such that  $\phi \circ \eta_0 = \eta_1|_F$ .
- (7) A symmetric homeomorphism between proper compact subsets of  $\partial_e \mathbb{L}$  extends to a symmetric affine homeomorphism of  $\mathbb{L}$ .
- (8) Suppose that  $H$  is a closed biface of  $\mathbb{L}$ . Consider the complementary biface  $H'$  endowed with the  $w^*$ -topology induced by the functions  $a \in A_\sigma(\mathbb{L})$  such that  $a|_H \equiv 0$ . Then  $H'$  is affinely homeomorphic to  $\mathbb{L}$ .

One can make (6) more precise, and assert that for any closed proper biface  $H$  of  $\mathbb{L}$  symmetrically affinely homeomorphic to  $\mathbb{L}$  the map  $\Omega_{\mathbb{G}} : A_\sigma(\mathbb{L}) \rightarrow A_\sigma(H)$ ,  $f \mapsto f|_H$  is (conjugate to) the universal nonexpansive operator on the Gurarij space constructed by Garbulińska-Wegryn and Kubiś in [45].

In §6.2 we also obtain the natural analogs of (1)–(6) above for *complex* Banach spaces. If  $X$  is a complex Banach space, then we regard the unit ball  $\text{Ball}(X^*)$  of the dual space of  $X$  as a compact circled convex set with a distinguished action of  $\mathbb{T}$  given by  $(\lambda, x) \mapsto \lambda x$ . A  $w^*$ -continuous affine map between compact circled convex sets is *homogeneous* if it commutes with such an action.

A complex Banach space  $X$  can be identified with the space  $A_{\mathbb{T}}(\text{Ball}(X^*))$  of complex-valued  $w^*$ -continuous homogeneous affine functions on  $\text{Ball}(X^*)$ . Furthermore, the map  $K \mapsto A_{\mathbb{T}}(K)$  is a contravariant equivalence of categories from the category of compact circled convex sets and homogeneous continuous maps to the category of complex Banach spaces and continuous linear maps of norm at most 1. The class of compact circled convex sets corresponding to complex Lindenstrauss spaces has been characterized by Effros in [34, Theorem 4.3]. We will refer to them as *Effros simplices*. The natural analog of a biface in this setting is the circled convex hull of a face (*circled face*); see Definition 6.16. We will note in §6.2 that statements (1)–(9) as in Theorem 1.2 hold for complex Banach spaces and Effros simplices, as long as bifaces are replaced with circled faces, Lazar simplices are replaced with Effros simplices, and symmetric functions are replaced with homogeneous functions.

We will develop in Section 7 an even more general framework to Fraïssé limits in functional analysis. The goal of this further generalization is to obtain the natural noncommutative analogs of the results above concerning the Gurarij space and the Poulsen simplex. It is well known since the groundbreaking work of Arveson [4, 5] that operator systems provide the natural noncommutative analog of compact convex sets. Indeed compact convex sets as discussed above correspond via the map  $K \mapsto A(K)$  to function systems, which are unital self-adjoint subspaces of unital abelian  $C^*$ -algebras. By replacing unital abelian  $C^*$ -algebras with arbitrary unital  $C^*$ -algebras one obtains the notion of an operator system.

Let  $B(H)$  be the algebra of bounded linear operators on a Hilbert space  $H$  endowed with the operator norm. Concretely, an operator system is a closed subspace  $X$  of  $B(H)$  that contains the identity operator 1 and is closed under taking adjoints. Abstractly, an operator system can be regarded as a complex vector space containing a distinguished element 1 (the *unit*) endowed with the following further structure: a function  $x \mapsto x^*$  (corresponding to taking adjoints) and a norm on the space  $M_n(X)$  on the space of  $n \times n$  matrices of elements of  $X$  inherited from the inclusion  $M_n(X) \subset M_n(B(H))$ . Here  $M_n(B(H))$  is endowed with the operator norm coming from the identification of  $M_n(B(H))$  with the space of operators on the Hilbertian sum of  $n$  copies of  $H$ . The corresponding notion of morphism  $\phi : X \rightarrow Y$  is a *unital completely contractive map*. This means that  $\phi$  maps the unit to the unit (*unital*), and  $\|\phi^{(n)}(x)\| \leq \|x\|$  for any  $n \in \mathbb{N}$  and  $x \in M_n(X)$  (*completely*

contractive), where  $\phi^{(n)}(x)$  is the element of  $M_n(Y)$  obtained from  $x$  by applying  $\phi$  entrywise. For a unital map, being completely contractive is equivalent to being completely positive, which amounts at requiring that  $\phi^{(n)}(x)$  is positive whenever  $x \in M_n(X)$  is positive. Any function system  $A(K)$  has a canonical (minimal) operator system structure, with matrix norms defined by  $\|x\| = \sup_{\phi} \|\phi^{(n)}(x)\|$  for  $x \in M_n(A(K))$ , where  $\phi$  ranges among all the unital positive linear functionals on  $A(K)$ .

Works of Effros [35], Wittstock [108], Effros and Winkler [40], Webster and Winkler [106], and Winkler [107], have made it clear that there exists a natural geometric object that corresponds to an operator system and completely encodes its structure: the matrix state space. If  $X$  is an operator system, let  $S_n(X)$  be the compact convex set of unital completely positive linear maps from  $X$  to  $M_n(\mathbb{C})$ . The *matrix state space*  $\mathbf{S}(X)$  of  $X$  is the sequence  $(S_n(X))_{n \in \mathbb{N}}$ . In  $\mathbf{S}(X)$  one can define the notion *matrix convex combination*, which is an expression of the form  $\gamma_1^* v_1 \gamma_1 + \dots + \gamma_\ell^* v_\ell \gamma_\ell$  for  $v_i \in S_{n_i}(X)$  and invertible  $\gamma_i \in M_{n, n_i}(\mathbb{C})$ . Such a matrix convex combination is *proper* if  $\gamma_i$  is right invertible for  $i = 1, 2, \dots, \ell$  and  $\gamma_1^* \gamma_1 + \dots + \gamma_\ell^* \gamma_\ell = 1$ , and *trivial* if for  $i = 1, 2, \dots, \ell$  there exist  $t_i \in [0, 1]$  such that  $\gamma_i^* \gamma_i = t_i 1$  and  $\gamma_i^* v_i \gamma_i = t_i v$  for  $i = 1, 2, \dots, n$ . An element of  $\mathbf{S}(X)$  is a *matrix extreme point* if it can not be written in a nontrivial way as a proper matrix convex combination (such a definition of matrix extreme point is equivalent to [106, Definition 2.1] in view of [42, Theorem A]). The original operator system  $X$  can be canonically identified with the space  $A(\mathbf{S}(X))$  of *matrix affine*  $w^*$ -continuous mappings from  $\mathbf{S}(X)$  to  $\mathbf{S}(\mathbb{C})$  [106, Definition 3.4].

Generally, a *compact matrix convex set*  $\mathbf{K}$  is a sequence  $(K_n)$  of compact convex sets  $K_n \subset M_n(V)$  for some locally convex topological vector space  $V$ , that is closed under matrix convex combinations [106, Definition 1.1]. Any compact matrix convex set arises from an operator system as described above [106, Proposition 3.5]. Furthermore the map  $\mathbf{K} \rightarrow A(K)$  is a contravariant equivalence of categories from the category of compact matrix convex sets and matrix affine continuous maps to the category of operator systems and unital completely positive maps.

An operator system  $A(\mathbf{K})$  is *nuclear* if the identity map of  $A(\mathbf{K})$  is the pointwise limit of unital completely positive maps that factor through finite-dimensional injective operator systems. In the commutative case, a function system  $A(K)$  is nuclear if and only if  $A(K)$  is a Lindenstrauss space, which is in turn equivalent to the assertion that  $K$  is a Choquet simplex; see [13, §8.6.4] and Subsection 6.3 below. Consistently, we say that a compact matrix convex set  $\mathbf{K}$  is a noncommutative Choquet simplex if the associated operator system  $A(\mathbf{K})$  is nuclear. Several characterizations of noncommutative Choquet simplices are established in [28], generalizing the Choquet-Meyer, Bishop-de Leeuw, and Namioka-Phelps characterization of Choquet simplices [11, 89, 95].

Suppose that  $F$  is a compact convex subset of a metrizable Choquet simplex  $K$ . It follows from works of Lazar [71] and Alfsen and Effros [2, 3, 33] that  $F$  is a face if and only if the map  $f \mapsto f|_F$  is a unital quotient mapping whose kernel is an  $M$ -ideal of  $A(K)$ ; see Proposition 6.21 below. We consider the noncommutative analog of such a notion, and call a compact matrix convex subset  $\mathbf{F}$  of a metrizable compact matrix convex set  $\mathbf{K}$  a *closed matrix face* if the canonical unital completely positive map  $A(\mathbf{K}) \rightarrow A(\mathbf{F})$  is a complete quotient mapping and its kernel is a complete  $M$ -ideal in the sense of Effros and Ruan [38].

The natural noncommutative analog  $\mathbf{NP}$  of the Poulsen simplex  $\mathbb{P}$  is the matrix state space of the Fraïssé limit  $A(\mathbf{NP})$  of the class of exact finite-dimensional operator systems. We will call  $\mathbf{NP}$  the *noncommutative Poulsen simplex* and  $A(\mathbf{NP})$  the *noncommutative Poulsen system*. A direct proof of existence and uniqueness of  $\mathbf{NP}$  can be found in [28]. It is also proved in [28] that  $\mathbf{NP}$  is the unique nontrivial metrizable noncommutative Choquet simplex with dense matrix extreme boundary, and  $A(\mathbf{NP})$  is the unique separable nuclear operator systems that is universal in the sense of Kirchberg and Wassermann [63]. The model-theoretic properties of the noncommutative Poulsen system have been investigated in [49]. The following noncommutative analog of Theorem 1.3 follows from our general results; see Subsection 8.2.

**Theorem 1.3.** *Let  $A(\mathbf{NP})$  be the Fraïssé limit of the class of finite-dimensional exact operator systems, and let  $\mathbf{NP}$  be its matrix state space.*

- (1)  *$A(\mathbf{NP})$  is a nuclear operator system, and it is the unique separable exact approximately homogeneous operator system that contains unital completely isometric copies of any separable exact operator system.*
- (2) *The set of matrix extreme points of  $\mathbf{NP}$  is dense.*
- (3) *A metrizable compact matrix convex set is a noncommutative Choquet simplex if and only if it is matrix affinely homeomorphic to a closed proper matrix face of  $\mathbf{NP}$ .*
- (4) *The canonical action of  $\text{Aut}(\mathbf{NP})$  on the state space of  $A(\mathbf{NP})$  is minimal.*

The noncommutative Poulsen system is, in particular, the first example of a nuclear operator system that contains a completely isometric copy of any separable exact operator system.

We also consider the noncommutative analogs of the Gurarij Banach space and of the Lusky simplex. Operator spaces [98, 39] are the noncommutative analog of (complex) Banach spaces. Indeed a Banach space can be seen as a subspace of an abelian  $C^*$ -algebra. By considering arbitrary, not necessarily abelian  $C^*$ -algebras, one obtains the notion of an operator space. Concretely, an operator space is a closed subspace of the algebra  $B(H)$  of bounded linear operators on a Hilbert space. Abstractly, an operator space  $X \subset B(H)$  can be seen as a structure consisting of the vector space operations together with the *matrix norms* arising from the inclusion  $M_n(X) \subset M_n(B(H))$ . The corresponding notion of morphism is a completely contractive linear map. An operator space is *nuclear* if the identity map of  $X$  is the pointwise limit of completely contractive maps that factor through finite-dimensional injective operator spaces.

Any Banach space can be regarded as an operator space with its canonical minimal operator space structure (minimal quantization); see [39, Section 3.3]. A Banach space  $X$  is Lindenstrauss if and only if it is a nuclear operator space with its minimal operator space structure [13, Proposition 8.6.5]. Thus, nuclear operator spaces can be seen as the noncommutative analog of Lindenstrauss spaces.

As in the case of Banach spaces, one can associate with an operator space a geometric object that completely encodes its structure. Suppose that  $X$  is an operator space. We let the complete dual ball  $\text{CBall}(X^*)$  to be the sequence  $(K_{n,m})_{n,m \in \mathbb{N}}$  where  $K_{n,m}$  is the unit ball of  $M_{n,m}(X^*)$ . It is easy to see that  $\text{CBall}(X^*)$  is closed under *rectangular matrix convex combinations*. These are expressions of the form  $\alpha_1^* v_1 \beta_1 + \cdots + \alpha_\ell^* v_\ell \beta_\ell$  where  $\alpha_i \in M_{n_i,n}(\mathbb{C})$  and  $\beta_i \in M_{m_i,m}$  and  $v_i \in K_{n_i m_i}$  for  $1 \leq i \leq \ell$ .

If  $V$  is a locally convex vector space, and  $\mathbf{K}$  is a collection of compact subsets  $K_{n,m} \subset M_{n,m}(V)$ , then we say that  $\mathbf{K}$  is a compact *rectangular matrix convex set* if it is closed under rectangular matrix convex combinations. The notions of matrix affine map, matrix affine combination, and matrix affine extreme point have natural rectangular analogs. The bipolar theorem and the Krein-Milman theorem for compact rectangular matrix convex sets have been established in [44, Section 3].

If  $\mathbf{K}$  is a compact rectangular convex set, then we let  $A_\sigma(\mathbf{K})$  be the space of continuous rectangular affine maps from  $\mathbf{K}$  to  $\mathbb{C}$  endowed with its canonical operator space structure; see [44, Section 3]. It is proved in [44, Section 3] using the bipolar theorem for compact rectangular matrix convex sets that if  $\mathbf{K}$  is a compact rectangular matrix convex set, then  $\mathbf{K}$  can be identified with  $\text{CBall}(X^*)$ , where  $X$  is the operator space  $A_\sigma(\mathbf{K})$ , via the map sending  $x \in X$  to the continuous rectangular affine map  $[\phi_{ij}] \mapsto [\phi_{ij}(x)]$ . Furthermore the assignment  $\mathbf{K} \mapsto A_\sigma(\mathbf{K})$  is a contravariant equivalence of categories from the category of compact rectangular matrix convex and continuous rectangular matrix convex maps to the category of operator spaces and completely contractive linear maps. Consistently with the commutative setting, say that  $\mathbf{K}$  is a (metrizable) noncommutative Lazar simplex if  $A_\sigma(\mathbf{K})$  is nuclear.

It has been proved by Lazar and Lindenstrauss that a compact absolutely convex subset  $F$  of a Lazar simplex  $\mathbf{K}$  is a closed biface if and only if the kernel of the map  $A_\sigma(\mathbf{K}) \rightarrow A_\sigma(F)$ ,  $f \mapsto f|_F$  is an  $M$ -ideal; see Proposition 6.5. A similar characterization holds for complex Lindenstrauss spaces by results of Ellis-Rao-Roy-Utterud [41] and Olsen [94]; see Proposition 6.17. Consistently, we define a closed rectangular matrix face of a compact rectangular matrix convex set  $\mathbf{K}$  to be a compact rectangular matrix convex subset  $\mathbf{F}$  of  $\mathbf{K}$  such that the map  $A_\sigma(\mathbf{K}) \rightarrow A_\sigma(\mathbf{F})$ ,  $f \mapsto f|_{\mathbf{F}}$  is a complete quotient mapping whose kernel is a complete  $M$ -ideal.

The natural noncommutative analog of the Gurarij space is a nuclear operator space that is approximately ultrahomogeneous and contains a completely isometric copy of any separable exact operator space. It follows from the general results of this paper that such a space exists, it is unique, and it coincides with the noncommutative Gurarij space  $\text{NG}$  defined in [91] and proved to be unique in [81]. The space  $\text{CBall}(\text{NG}^*)$  can be seen as the noncommutative analog of the Lusky simplex  $\mathbb{L} = \text{Ball}(\mathbb{G}^*)$ . We will call  $\text{CBall}(\text{NG}^*)$  the *noncommutative Lusky simplex* and denote it by  $\text{NL}$ . The following result is the natural noncommutative analog of (the complex version of) Theorem 1.2.

**Theorem 1.4.** *Let  $\text{NG}$  be the Gurarij operator space, and  $\text{NL}$  be the noncommutative Lazar simplex.*

- (1)  *$\text{NG}$  is a nuclear operator space, and it is the unique separable exact operator space that contains an isometric copy of any separable exact operator space [81].*
- (2) *The set of rectangular matrix extreme points of  $\text{NL}$  is dense in  $\text{NL}$ .*
- (3) *A metrizable compact rectangular convex set is a noncommutative Lazar simplex if and only if it is rectangular affinely homeomorphic to a proper closed rectangular matrix face of  $\text{NL}$ .*

Our framework also covers other new examples of Fraïssé classes, such as the class of finite-dimensional operator sequence spaces (§6.5), and the class of finite-dimensional  $p$ -multinormed spaces for every  $p \in (1, +\infty)$  (§6.4). The corresponding limits  $\mathbb{CG}$  (the *column Gurarij space*) and  $\mathbb{GM}^p$  (the  *$p$ -multinormed Gurarij space*) give new examples—in addition to  $\mathbb{G}$  and  $A(\mathbb{P})$ —of separable metric structures whose first order theory is separably

categorical and admits elimination of quantifiers [8, §13]. As a consequence the corresponding automorphism groups  $\text{Aut}(\mathbb{G})$ ,  $\text{Aut}(\mathbb{P})$ ,  $\text{Aut}(\mathbb{CG})$ , and  $\text{Aut}(\mathbb{GM}^p)$  for  $p \in (1, +\infty)$  are new examples of Roelcke precompact Polish groups [10, Definition 1.1, Theorem 2.4]. Similar results as the ones mentioned above hold for  $\mathbb{CG}$  and  $\mathbb{GM}^p$ .

In addition to the results above, our general framework will apply to produce commutative and noncommutative analogs of universal operators in the sense of Rota, generalizing work of Garbulińska-Węgrzyn and Kubiś [45]; see Theorem 4.1, Theorem 4.2, Theorem 4.3, Theorem 4.4, Theorem 5.1, Theorem 5.2, Theorem 5.3, and Theorem 5.5.

The rest of the paper is organized as follows. In Section 2 we present the general framework of Fraïssé classes generated by injective objects, and provide a characterization of the corresponding limits. A characterization of retracts of the limit  $M$  is provided in Section 3. The existence of generic (universal) morphisms  $M \rightarrow M$  in this general setting is proved in Section 4, while in Section 5 we prove the existence of generic morphisms  $M \rightarrow R$  for any separable approximately injective structure  $R$ . Section 6 provides several examples, explaining how real and complex Banach spaces, function systems,  $M_q$ -spaces,  $M_q$ -systems, operator sequence spaces, and  $p$ -multinormed spaces fit into the general framework. Finally Section 7 considers an even more general approach, suitable to deal with the cases of exact operator spaces and exact operator systems. These examples are presented in Section 8.

**Acknowledgments.** We would like to thank Itai Ben Yaacov, Ken Davidson, Isaac Goldbring, Ilijas Farah, Adam Fuller, Alexander Kechris, Matthew Kennedy, Michael Hartz, Ward Henson, Fernando Lledó, Jordi López-Abad, Wiesław Kubiś, Timur Oikhberg, Sławomir Solecki, Pedro Tradacete, and Todor Tsankov for their comments and many helpful conversations.

## 2. FRAÏSSÉ CLASSES GENERATED BY INJECTIVE OBJECTS

**2.1. Morphisms and embeddings.** Throughout this section we suppose that  $\mathcal{L}$  is a countable *language* in the logic for metric structures. For simplicity we will assume that  $\mathcal{L}$  is single-sorted. A complete introduction to the logic for metric structures can be found in [8]. We recall here the key concepts. The language  $\mathcal{L}$  is a countable collection of *function symbols* and *relation symbols*. Every symbol  $B$  in  $\mathcal{L}$  has assigned an *arity*  $n_B \in \mathbb{N}$  and a *modulus of continuity*  $\varpi_B$ . An  $\mathcal{L}$ -structure  $X$  is a complete metric space with metric bounded by 1 endowed with the *interpretation*  $X^B$  for any relation symbol  $B$  in  $\mathcal{L}$ . Here  $X^B$  is a function from  $X^{n_B}$  to either  $X$  or a compact interval  $[\lambda_B, \mu_B] \subset \mathbb{R}$  (depending whether  $B$  is a function or a relation symbol) that is uniformly continuous with modulus  $\varpi_B$  with respect to the supremum metric on  $X^{n_B}$ . We will assume that  $\mathcal{L}$  contains a distinguished binary relation symbol whose interpretation in an  $\mathcal{L}$ -structure is the distance function.

Suppose that  $(x_n)$  is a fixed collection of *variables*. We denote by  $\bar{x}$  a tuple of such variables. Terms in the language  $\mathcal{L}$  are defined recursively by declaring that any variable  $x$  is a term  $t(x)$ , and if  $t_1(\bar{x}_1), \dots, t_n(\bar{x}_n)$  are terms, and  $f$  is an  $n$ -ary function symbol in  $\mathcal{L}$ , then  $f(t_1, \dots, t_n)$  is a term  $t(\bar{x}_1, \dots, \bar{x}_n)$ . An atomic formula  $\varphi(\bar{x})$  in the language  $\mathcal{L}$  is an expression of the form  $B(t_1(x), \dots, t_n(x))$  where  $t_1, \dots, t_n$  are terms and  $B$  is an  $n$ -ary relation symbol in  $\mathcal{L}$ . The interpretation of an atomic formula  $\varphi(x)$  in an  $\mathcal{L}$ -structure  $M$  is defined in the obvious way in terms of the interpretation of  $B$  and of the function symbols that appear in the terms  $t_1, \dots, t_n$ . A quantifier-free formula is an expression  $q(\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x}))$  where  $\varphi_1(\bar{x}), \dots, \varphi_n(\bar{x})$  are quantifier-free formulas and  $q : \mathbb{R}^n \rightarrow \mathbb{R}$  is a continuous function.

**Definition 2.1.** If  $E, F$  are  $\mathcal{L}$ -structures and  $T : E \rightarrow F$  is a function, then we say that  $T$  is:

- a *morphism* if  $T(\varphi(\bar{a})) \leq \varphi(\bar{a})$  for any atomic formula  $\varphi(x)$  and tuple  $\bar{a}$  in  $E$ ;
- an *embedding* if  $T(\varphi(\bar{a})) = \varphi(\bar{a})$  for any atomic formula  $\varphi(x)$  and tuple  $\bar{a}$  in  $E$ .

A *retraction* of an  $\mathcal{L}$ -structure  $A$  is a morphism  $r : A \rightarrow A$  such that  $r \circ r = r$ . A *retract* is the range of a retraction.

We regard  $\mathcal{L}$ -structures as objects of a category where morphisms are defined as in Definition 2.1. Observe that the isomorphisms in such a category are precisely the surjective embeddings. We note here that when these notions are applied to Banach spaces as metric structures (by identifying them with their unit ball), morphisms as in Definition 2.1 correspond to linear maps of norm at most 1, embeddings as in Definition 2.1 correspond to isometric linear maps, and isomorphisms as in Definition 2.1 correspond to linear isometric isomorphism.

**Definition 2.2.** If  $E$  is an  $\mathcal{L}$ -structure and  $\bar{a}$  is a finite tuple in  $E$ , then we denote by  $\langle \bar{a} \rangle$  the substructure of  $E$  generated by  $\bar{a}$ . This is by definition the set of  $b \in E$  such that, whenever  $f, g : E \rightarrow F$  are morphisms such that  $f(\bar{a}) = g(\bar{a})$ , one has that  $f(b) = g(b)$ . We say that  $X$  is *finitely generated* if  $X = \langle \bar{a} \rangle$  for some finite tuple  $\bar{a}$  in  $X$ . A subset  $Y$  of  $X$  is a *substructure* if it contains  $\langle \bar{a} \rangle$  for any finite tuple  $\bar{a}$  in  $Y$ .

The phrasing of the notion of substructure is chosen in such a way that, when a Banach space is seen as a structure by looking at its unit ball, then the substructure generated by a tuple  $\bar{a}$  coincides with the unit ball of the linear span of  $\bar{a}$ ; see Subsection 6.1.

Observe that if  $\phi : E \rightarrow F$  is a morphism, then the image  $\phi[E]$  of  $E$  under  $\phi$  is a substructure of  $F$ . If  $\bar{a}, \bar{a}'$  are two tuples in  $E$  of the same length, then we set  $d(\bar{a}, \bar{a}') = \max_i d(a_i, a'_i)$ . We convene that  $d(\bar{a}, \bar{a}') = +\infty$  if  $\bar{a}$  and  $\bar{a}'$  have different lengths.

**Definition 2.3.** If  $T, S : X \rightarrow Y$  are morphisms, then we let  $I(T)$  be the supremum of

$$|\varphi(\bar{a}) - \varphi(T(\bar{a}))| \quad (1)$$

where  $\varphi(x)$  is an atomic formula and  $\bar{a}$  is a tuple in  $X$ . Similarly we let  $d(T, S)$  be the supremum of  $d(T(x), S(x))$  where  $x$  ranges in  $X$ .

Observe that  $I(\phi \circ \psi) \leq I(\phi) + I(\psi)$ .

**Definition 2.4.** We define the Gromov-Hausdorff (GH) distance  $d(X, Y)$  of two structures  $X, Y$  in  $\mathcal{A}$  as follows:  $d(X, Y)$  is the infimum of  $\varepsilon > 0$  such that there exists morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that  $d(g \circ f, id_X) < \varepsilon$ ,  $d(f \circ g, id_Y) < \varepsilon$ ,  $I(f) < \varepsilon$ , and  $I(g) < \varepsilon$ .

It is not difficult to verify that the GH distance is indeed a metric. Dropping the requirement that  $I(f) < \varepsilon$  and  $I(g) < \varepsilon$  in Definition 2.4 yields an equivalent metric.

**2.2. Basic sequences.** Suppose that  $\mathcal{A}$  is a class of  $\mathcal{L}$ -structures such that

- (1) a structure belongs to  $\mathcal{A}$  if and only if each of its finitely generated substructures belong to  $\mathcal{A}$ ,
- (2)  $\mathcal{A}$  is closed under inductive limits with embeddings as connective maps,
- (3)  $\mathcal{A}$  has arbitrary products,
- (4)  $\mathcal{A}$  has a universal initial object, which is a finitely generated structure,
- (5) if  $f_i : X \rightarrow Y_i$  is a collection of morphism between structures in  $\mathcal{A}$ ,  $Y$  is the product of  $Y_i$ ,  $f : X \rightarrow Y$  is the morphism obtained from the universal property of the product,  $\varphi$  is an atomic formula, and  $\bar{a}$  is a tuple in  $X$ , then  $\varphi(f(\bar{a})) = \sup_i \varphi(f_i(\bar{a}))$ ,
- (6) for structures  $A, X, Y$  in  $\mathcal{A}$ , morphisms  $f_X^{(i)} : A \rightarrow X$  and  $f_Y^{(i)} : A \rightarrow Y$  for  $i = 1, 2, \dots, n$ , atomic formula  $\varphi(x)$ , and tuple  $\bar{a}$  in  $A$ , if  $Z$  is the product of  $X$  and  $Y$ ,  $f^{(i)} : A \rightarrow Z$  are the morphisms obtained from  $f_X^{(i)}$  and  $f_Y^{(i)}$ , respectively, and the universal property of the product, then  $\varphi(f^{(1)}(\bar{a}), \dots, f^{(n)}(\bar{a})) \leq \max\{\varphi(f_X^{(1)}(\bar{a}), \dots, f_X^{(n)}(\bar{a})), \varphi(f_Y^{(1)}(\bar{a}), \dots, f_Y^{(n)}(\bar{a}))\}$ ;
- (7) if  $A, B$  are finitely-generated structures in  $\mathcal{A}$ , then the space of morphisms from  $A$  to  $B$  is totally bounded with respect to the metric from Definition 2.3.

Observe that in particular these assumptions guarantee that the canonical morphism from  $X$  to the product of  $X$  and  $Y$  is an embedding. We also suppose that any structure  $X$  in  $\mathcal{A}$  is endowed with a collection of finite tuples of pairwise distinct elements of  $X$  that we call *basic tuples*. We assume that any finitely generated structure in  $\mathcal{A}$  has a generating basic tuple, and any finite tuple contains a basic subtuple.

**Definition 2.5.** We say that a subset  $D$  of a structure  $X$  in  $\mathcal{A}$  is fundamental if it generates a dense substructure of  $X$ , and the set of basic tuples from  $D$  is dense in the set of basic tuples from  $X$ .

We assume that any separable structure in  $\mathcal{A}$  has a countable fundamental subset. In the following we fix for every separable structure  $X$  in  $\mathcal{A}$  a countable fundamental subset  $D_X$  of  $X$ . We also assume that for any structure  $X$  in  $\mathcal{A}$  and basic tuple  $\bar{a}$  in  $X$  there exists a strictly increasing function  $\rho_{\bar{a}} : [0, \delta_{\bar{a}}) \rightarrow [0, +\infty)$  that is vanishing at 0 and continuous at 0 such that if  $f, g : X \rightarrow Y$  are morphisms such that  $d(f(\bar{a}), g(\bar{a})) \leq \delta \leq \delta_{\bar{a}}$ , then there exists a morphism  $h : \langle f(\bar{a}) \rangle \rightarrow Y$  such that  $d(h \circ f, g) \leq \rho_{\bar{a}}(\delta)$ . The latter requirement can be seen as the assertion that basic tuples satisfy the natural analogue of the small perturbation lemma from Banach space and operator space theory [98, Lemma 2.13.2].

A *marked structure*  $(E, \bar{a})$  in  $\mathcal{A}$  is a structure  $E$  in  $\mathcal{A}$  endowed with a distinguished generating basic tuple  $\bar{a}$ . We call a marked structure  $(E, \bar{a})$  where  $\bar{a}$  has length  $n$  an  $n$ -marked structure. In the following we denote the marked structure  $(E, \bar{a})$  simply by  $\bar{a}$  and refer to  $E$  as  $\langle \bar{a} \rangle$ . If  $\bar{a}, \bar{b}$  are  $n$ -marked structures, we let  $\partial(\bar{a}, \bar{b})$  be the infimum of  $\max\{I(f), d(f(\bar{a}), \bar{b})\}$  where  $f$  ranges among all the morphisms  $f : \langle \bar{a} \rangle \rightarrow \langle \bar{b} \rangle$ . Observe that  $\partial(\bar{a}, \bar{c}) \leq \partial(\bar{a}, \bar{b}) + \partial(\bar{b}, \bar{c})$ . However  $\partial$  might not be symmetric, and hence it is not a metric in general.

**2.3. Fraïssé classes generated by injective objects.** We say that a structure  $A$  in  $\mathcal{A}$  is *injective* if it is an injective object of  $\mathcal{A}$  when regarded as a category with the notion of morphisms from Definition 2.1. This means that if  $X \subset Y$  are structures in  $\mathcal{A}$  and  $f : X \rightarrow A$  is a morphism, then there exists a morphism  $g : Y \rightarrow A$  that extends  $f$ . We suppose in the following that  $\mathcal{I}$  is a countable collection of finitely generated injective elements of  $\mathcal{A}$  closed under finite products.

For now and the rest of the section we fix a function  $\varpi : [0, +\infty) \rightarrow [0, +\infty)$  that is a strictly increasing, continuous at 0, and vanishing at 0.

**Definition 2.6.** *The class  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  with modulus  $\varpi$  if every finitely-generated structure in  $\mathcal{A}$  is the limit with respect to the Gromov-Hausdorff distance of finitely-generated substructures of structures in  $\mathcal{I}$ , and for any separable structures  $X, \widehat{X}, A$  in  $\mathcal{A}$  with  $X$  finitely generated and  $A \in \mathcal{I}$ , and morphisms  $\phi : X \rightarrow \widehat{X}$  and  $f : X \rightarrow A$  such that  $I(\phi) \leq \delta$  there exists a morphism  $h : \widehat{X} \rightarrow A$  such that  $d(h \circ \phi, f) \leq \varpi(\delta)$ .*

**Definition 2.7.** *A structure  $M$  in  $\mathcal{A}$  is stably homogeneous with modulus  $\varpi$  if whenever  $E$  is a finitely generated structure in  $\mathcal{A}$ , and  $\phi : E \rightarrow M$  and  $f : E \rightarrow M$  are morphisms such that  $I(f) < \delta$  and  $I(\phi) < \delta$ , then there exists an automorphism  $\alpha$  of  $M$  such that  $d(\alpha \circ \phi, f) < \varpi(\delta)$ .*

The following is the main general theorem characterizing Fraïssé classes generated by injective objects. We will recall the notion of (metric) Fraïssé class as defined in [7, Definition 3.12] in Subsection 2.5.

**Theorem 2.8.** *Assume that  $\mathcal{A}$  is a category of  $\mathcal{L}$ -structures satisfying the assumptions of Subsection 2.2. Let  $\mathcal{I}$  be a collection of finitely generated injective structures of  $\mathcal{A}$  closed under finite products. Denote by  $\mathcal{C}$  the class of finitely-generated structures in  $\mathcal{A}$ . The following statements are equivalent:*

- (1)  *$\mathcal{C}$  is a Fraïssé class, the limit  $M$  of  $\mathcal{C}$  can be realized as an inductive limit of structures from  $\mathcal{I}$  with embeddings as morphisms, any structure in  $\mathcal{I}$  is isomorphic to a retract of  $M$ , and  $M$  is stably homogeneous with modulus  $\varpi$ ;*
- (2)  *$\mathcal{A}$  has enough injectives from  $\mathcal{I}$  with modulus  $\varpi$ .*

The implication (1) $\Rightarrow$ (2) is a consequence of the universality property of the Fraïssé limit together with our assumption on basic sequences. The rest of this section and the next section are devoted to prove the implication (2) $\Rightarrow$ (1). We will assume throughout that  $\mathcal{A}$  and  $\mathcal{I}$  are classes of  $\mathcal{L}$ -structures satisfying the assumptions of Theorem 2.8 and such that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  with modulus  $\varpi$ . A characterization of the Fraïssé limit of  $\mathcal{C}$  will be given in Proposition 2.12.

**2.4. Approximate pushouts.** In this subsection we prove that the assumptions above on  $\mathcal{A}$  allow one to *amalgamate* the structures in  $\mathcal{A}$  over a common substructure.

**Lemma 2.9.** *Suppose that  $E, X, Y$  are separable structures in  $\mathcal{A}$  such that  $X, Y$  belong to  $\mathcal{I}$  and  $E$  is finitely generated, and  $f_X : E \rightarrow X$  and  $f_Y : E \rightarrow Y$  are morphisms. If  $I(f_X) \leq \delta$  and  $I(f_Y) \leq \delta$ , then there exists a structure  $Z$  in  $\mathcal{I}$  and embeddings  $i : X \rightarrow Z$  and  $j : Y \rightarrow Z$  such that  $d(i \circ f_X, j \circ f_Y) \leq \varpi(\delta)$ .*

*Proof.* Since  $Y$  is injective, and  $\mathcal{A}$  enough injectives from  $\mathcal{I}$  with modulus  $\varpi$ , there exists a morphism  $h_X : X \rightarrow Y$  such that  $d(h_X \circ f_X, f_Y) \leq \varpi(\delta)$ . Similarly there exists a morphism  $h_Y : Y \rightarrow X$  such that  $d(h_Y \circ f_Y, f_X) \leq \varpi(\delta)$ . Let now  $Z$  be the product of  $X$  and  $Y$ , and  $i : X \rightarrow Z$  be the morphism obtained from the morphisms  $id_X : X \rightarrow X$  and  $h_X : X \rightarrow Y$  using the universal property of the product. Similarly let  $j : Y \rightarrow Z$  be the morphism obtained from the morphisms  $h_Y : Y \rightarrow X$  and  $id_Y : Y \rightarrow Y$  using the universal property of the product. Observe that  $i$  and  $j$  are embeddings. Furthermore  $d(i \circ f_X, j \circ f_Y) \leq \varpi(\delta)$  by Condition (6) of Subsection 2.2.  $\square$

**Lemma 2.10.** *Suppose that  $X, \widehat{X}, Y$  are structures in  $\mathcal{A}$ , and  $\phi : X \rightarrow \widehat{X}$  and  $f : X \rightarrow Y$  are morphisms such that  $I(\phi) \leq \delta$ . Then there exist a structure  $\widehat{Y}$  in  $\mathcal{A}$ , a morphism  $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$ , and an embedding  $j : Y \rightarrow \widehat{Y}$  such that  $d(\widehat{f} \circ \phi, j \circ f) \leq \varpi(\delta)$  and furthermore for any structure  $Z$  in  $\mathcal{A}$  and morphisms  $g : \widehat{X} \rightarrow Z$  and  $h : Y \rightarrow Z$  such that  $d(g \circ \phi, h \circ f) \leq \varpi(\delta)$  there exists a morphism  $\tau : \widehat{Y} \rightarrow Z$  such that  $g = \tau \circ \widehat{f}$  and  $h = \tau \circ j$ . If moreover  $I(f) \leq \delta$ , then  $f$  is an embedding. If  $\widehat{X}, Y$  are finitely generated, then  $\widehat{Y}$  is finitely generated.*

*Proof.* Consider the collection  $(g_i, h_i)$  of all the morphisms  $g_i : \widehat{X} \rightarrow A_i$  and  $h_i : Y \rightarrow A_i$  for  $A_i \in \mathcal{I}$  such that  $d(g_i \circ \phi, h_i \circ f) \leq \varpi(\delta)$ . Let  $W$  be the product of  $A_i$  in  $\mathcal{A}$  and  $\widehat{f} : \widehat{X} \rightarrow W$  and  $j : Y \rightarrow W$  the morphisms obtained from the morphisms  $g_i$  and  $h_i$  and the universal property of the product. We claim that  $j$  is an embedding. In fact, suppose that  $\bar{b}$  is a tuple in  $Y$  and  $\psi(x)$  is an atomic formula such that  $\psi(\bar{b}) > r$ . Then there exists  $A \in \mathcal{I}$  and a morphism  $h : Y \rightarrow A$  such that  $\psi(h(\bar{b})) > r$ . Since  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$ ,



there exists a morphism  $g : \widehat{X} \rightarrow A$  such that  $d(g \circ \phi, h \circ f) \leq \varpi(\delta)$ . Therefore  $g = g_i$  and  $h = h_i$  for some  $i$  as above and hence

$$\psi(j(\bar{b})) \geq \psi(h(\bar{b})) > r.$$

This shows that  $j : Y \rightarrow W$  is an embedding. Let  $\widehat{Y}$  be the substructure of  $W$  generated by the union of the ranges of  $\widehat{f}$  and  $j$ . Suppose now that  $Z$  is a structure in  $\mathcal{A}$  and  $g : \widehat{X} \rightarrow Z$  and  $\eta : \widehat{Y} \rightarrow Z$  are morphisms such that  $d(g \circ \phi, \eta \circ f) \leq \varpi(\delta)$ . Since  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$ ,  $Z$  embeds into a product  $\widehat{Z}$  of structures in  $\mathcal{I}$ . The definition of  $W$  above guarantees the existence of a unique morphism  $\tau : W \rightarrow \widehat{Z}$  such that  $\tau \circ \widehat{f} = g$  and  $\tau \circ j = \eta$ . Since  $\widehat{Y}$  is the substructure of  $W$  generated by the ranges of  $\widehat{f}$  and  $j$ , we have that  $\tau$  maps  $\widehat{Y}$  into  $Z$ . Finally under the assumption that  $I(f) \leq \delta$  one can prove that  $\widehat{f}$  is an embedding reasoning as above.  $\square$

The structure  $\widehat{Y}$  in  $\mathcal{A}$  constructed in Lemma 2.10 will be called the *approximate pushout* of the morphisms  $f$  and  $\phi$  with tolerance  $\varpi(\delta)$ .

**2.5. The Fraïssé class.** Let  $\mathcal{C}$  be the class of finitely generated elements of  $\mathcal{A}$ . We aim at showing that  $\mathcal{C}$  is a (complete) Fraïssé class in the sense of [7, Definition 3.12]. Fix  $n \in \mathbb{N}$  and let  $\mathcal{C}_n$  be the class of  $n$ -marked structures in  $\mathcal{A}$ . (It should be remarked that arbitrary tuples of generators are considered in [7], rather than only basic tuples as we do here. However this does not pose any problem, and all the results in [7] go through only considering basic tuples.) Recall that the Fraïssé metric  $d_{\mathcal{C}}$  on  $\mathcal{C}_n$  is defined by

$$d_{\mathcal{C}}(\bar{a}, \bar{b}) = \inf_{\phi, \psi} d(\phi(\bar{a}), \psi(\bar{b}))$$

where  $\phi : \langle \bar{a} \rangle \rightarrow Z$  and  $\psi : \langle \bar{b} \rangle \rightarrow Z$  range among all the joint embeddings into a third structure  $Z$  in  $\mathcal{C}$ ; see also [7, Definition 3.11].

In order to prove that  $\mathcal{C}$  is a Fraïssé class as in [7, Definition 3.12], we need to show that

- $\mathcal{C}$  satisfies the *hereditary property* (HP), that is,  $\mathcal{C}$  is closed under taking finitely generated substructures;
- $\mathcal{C}$  satisfies the *joint embedding property* (JEP), that is, any two structures in  $\mathcal{C}$  simultaneously embed into a third structure in  $\mathcal{C}$ ;
- $\mathcal{C}$  satisfies the *near amalgamation property* (NAP), that is, if  $\bar{a}$  is a marked structure in  $\mathcal{C}$ ,  $\varepsilon > 0$ ,  $B_i$  are structures in  $\mathcal{C}$  and  $\phi_i : \langle \bar{a} \rangle \rightarrow B_i$  are embeddings for  $i \in \{0, 1\}$ , then there exists a structure in  $\mathcal{C}$  and embeddings  $\psi_i : B_i \rightarrow C$  such that  $d((\psi_0 \circ \phi_0)(\bar{a}), (\psi_1 \circ \phi_1)(\bar{a})) < \varepsilon$ ;
- $(\mathcal{C}_n, d_{\mathcal{C}})$  is a separable and complete metric space for every  $n \in \mathbb{N}$ .

Since  $\mathcal{A}$  is by assumption closed under substructures,  $\mathcal{C}$  satisfies the hereditary property. The joint embedding property is proved by taking binary products. Lemma 2.10 shows that  $\mathcal{C}$  satisfies the near amalgamation property. To conclude the proof it remains to show that  $(\mathcal{C}_n, d_{\mathcal{C}})$  is a separable and complete metric space.

Suppose that  $\bar{a}, \bar{b}$  are  $n$ -marked structures in  $\mathcal{A}$ . Recall that  $\partial(\bar{a}, \bar{b})$  is by definition

$$\inf_f \max \{I(f), d(f(\bar{a}), f(\bar{b}))\}$$

where  $f$  ranges among all the morphisms  $f : \langle \bar{a} \rangle \rightarrow \langle \bar{b} \rangle$ . It follows from Lemma 2.10 that

$$d_{\mathcal{C}}(\bar{a}, \bar{b}) \leq \varpi(\partial(\bar{a}, \bar{b})) + \partial(\bar{a}, \bar{b}). \quad (2)$$

Furthermore it follows from the assumptions on basic tuples from Subsection 2.2 that

$$\partial(\bar{a}, \bar{b}) \leq \rho_{\bar{a}}(d_{\mathcal{C}}(\bar{a}, \bar{b})). \quad (3)$$

Let  $(A_i)$  be an enumeration of the structures in  $\mathcal{I}$ . For any  $i \in \mathbb{N}$  let  $D_i \subset A_i$  be a countable fundamental subset; see Definition 2.5. Let  $(\bar{a}_{i,k})$  be an enumeration of all the basic  $n$ -tuples in  $D_i$ . It follows from the fact that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$ , Lemma 2.10, and our assumptions on basic tuples that if  $\bar{b}$  is an  $n$ -marked structure in  $\mathcal{A}$  and  $\varepsilon > 0$  then there exist  $i, k \in \mathbb{N}$  such that  $\partial(\bar{b}, \bar{a}_{i,k}) < \varepsilon$ . Together with Equation (2) this shows that  $\{\bar{a}_{i,k} : i, k \in \mathbb{N}\}$  is dense in  $(\mathcal{C}_n, d_{\mathcal{C}})$ .

Suppose now that  $(\bar{a}_j)$  is a Cauchy sequence in  $(\mathcal{C}_n, d_{\mathcal{C}})$ . Using Lemma 2.10 and the fact that  $\mathcal{A}$  is closed under limits of direct sequences with embeddings as connective maps one can show that there exists a structure  $X$  in  $\mathcal{A}$  and embeddings  $\phi_j : \langle \bar{a}_j \rangle \rightarrow X$  such that  $(\phi_j(\bar{a}_j))$  is a Cauchy sequence in  $X^n$  with max distance. If  $\bar{a}$  is a limit of such a sequence in  $X^n$ , then it is clear that  $\bar{a}$  is a limit of  $(\bar{a}_j)$  in  $(\mathcal{C}_n, d_{\mathcal{C}})$ . This concludes the proof that  $(\mathcal{C}_n, d_{\mathcal{C}})$  is complete, and  $\mathcal{C}$  is a Fraïssé class. In the following subsections we will give an independent proof of existence and uniqueness of the Fraïssé limit of  $\mathcal{C}$  in the sense of [7, Definition 3.15]; see also [7, Corollary 3.20].

It follows from the fact that  $(\mathcal{C}_n, d_{\mathcal{C}})$  is separable and our assumptions on basic sequences that the class of finitely generated structures in  $\mathcal{A}$  is separable with respect to the Gromov-Hausdorff distance introduced in Definition 2.4.

**2.6. Fraïssé limit: existence.** Here we want to give a direct proof—not relying on the general results from [7]—of existence of the limit of the class  $\mathcal{C}$  of finitely generated structures in  $\mathcal{A}$ . Precisely we will prove that there exists a separable structure  $M$  in  $\mathcal{A}$  that satisfies the following *approximate extension property* with modulus  $\varpi$ : if  $E = \langle \bar{a} \rangle$  and  $F$  are finitely generated structures in  $\mathcal{A}$ ,  $\varepsilon > 0$ ,  $\phi : E \rightarrow F$  and  $f : E \rightarrow M$  are morphisms such that  $\max\{I(\phi), I(f)\} < \delta$ , then there exists a morphism  $g : F \rightarrow M$  such that  $I(g) < \varepsilon$  and  $d(g \circ \phi, f) < \varpi(\delta)$ . It is easy to see using [7, Corollary 3.20] that a structure  $M$  satisfying the approximate extension property is a limit of  $\mathcal{C}$  in the sense of [7, Definition 3.15]. Furthermore the proof will show that  $M$  can be realized as the limit of an inductive sequence of elements of  $\mathcal{I}$  with embeddings as connective maps.

Let us say that a subset  $D$  of a metric space  $A$  is  $\varepsilon$ -dense for some  $\varepsilon > 0$  if every element of  $A$  is at distance at most  $\varepsilon$  from some element of  $D$ . Let  $(X_m)$  be a sequence of finitely generated structures in  $\mathcal{A}$  that is dense with respect to the Gromov-Hausdorff distance. Let  $(A_d)$  be an enumeration of the structures in  $\mathcal{I}$ . For every  $m, d, k \in \mathbb{N}$  let  $\mathcal{E}_{m,d,k}$  be a finite  $2^{-k}$ -dense set of morphisms from  $X_m$  to  $A_d$ . Using Lemma 2.9 one can define by recursion on  $k \in \mathbb{N}$  sequences  $(d_k), (j_k), (\mathcal{F}_{m,k})$  such that

- (1)  $d_k \in \mathbb{N}$ ,
- (2)  $j_k : A_{d_k} \rightarrow A_{d_{k+1}}$  is an embedding, and
- (3)  $\mathcal{F}_{m,k}$  is a finite  $2^{-k}$ -dense subset of the space of morphisms from  $X_m$  to  $A_{d_k}$ ,

such that for every  $m, d \leq k$ ,  $f \in \mathcal{F}_{m,k}$ , and  $\phi \in \mathcal{E}_{m,d,k}$  there exists  $\hat{f} : A_d \rightarrow A_{d_{k+1}}$  such that  $d(\hat{f} \circ \phi, j_k \circ f) \leq \varpi(\max\{I(f), I(\phi)\})$ .

One can define now  $M$  to be the limit of the inductive sequence  $(A_{d_k})$  with connective maps  $j_k : A_{d_k} \rightarrow A_{d_{k+1}}$ . It is not difficult to verify that  $M$  satisfies the approximate extension property using the assumption that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  together with our hypotheses on basic sequences.

**2.7. Fraïssé limit: uniqueness and stable homogeneity.** In this section we want to prove that the Fraïssé limit  $M$  of the class of finitely generated structures in  $\mathcal{A}$  is stably homogeneous with modulus  $\varpi$  in the sense of Definition 2.7. The argument is analogous to the one of [68, Theorem 1.1].

**Proposition 2.11.** *Let  $M$  be the limit of the class of finitely generated structures in  $\mathcal{A}$  as constructed in Subsection 2.6. Suppose that  $E$  is a finitely generated structure in  $\mathcal{A}$ ,  $\phi : E \rightarrow M$  and  $f : E \rightarrow M$  are morphisms such that  $I(f) < \delta$  and  $I(\phi) < \delta$ . Then there exists an automorphism  $\alpha$  of  $M$  such that  $d(\alpha \circ \phi, f) < \varpi(\delta)$ .*

Fix  $\eta, \delta_0 > 0$  such that  $\varpi(\delta_0) + \eta < \varpi(\delta)$ ,  $I(f) < \delta_0$ , and  $I(\phi) < \delta_0$ . Using the property of  $M$  established in Subsection 2.6 one can easily define by recursion on  $n$  increasing sequences  $(X_n)$  and  $(Y_n)$  of substructures of  $M$  with dense union,  $\delta_n > 0$ , morphisms  $\alpha_n : X_n \rightarrow Y_n$  and  $\beta_n : Y_n \rightarrow X_{n+1}$ , such that

- (1)  $X_1 \supset \phi[E]$ ,  $Y_1 \supset f[E]$ , and  $d(\alpha_1 \circ \phi, f) < \delta_0$ ,
- (2)  $\varpi(\delta_n) < 2^{-(n+1)}\eta$ ,
- (3)  $I(\alpha_n) < \delta_n$  and  $I(\beta_n) < \delta_n$ ,
- (4)  $d(\alpha_{n+1} \circ \beta_n, i_{Y_n}) < \varpi(\delta_n)$  and  $d(\beta_n \circ \alpha_n, i_{X_n}) < \varpi(\delta_n)$ .

In (4)  $i_{X_n}$  denotes the inclusion map of  $X_n$  into  $X$ , and  $i_{Y_n}$  denotes the inclusion map from  $Y_n$  into  $Y$ . It then follows from (3) and (4) that

$$d(\alpha_n, (\alpha_{n+1})|_{X_n}) < 2\varpi(\delta_n) < 2^{-n}\eta$$

and similarly for  $\beta_n$  and  $(\beta_{n+1})|_{Y_n}$ . Therefore the sequences  $(\alpha_n)$  and  $(\beta_n)$  induce morphisms  $\alpha : M \rightarrow M$  and  $\beta : M \rightarrow M$ . By (3)  $\alpha$  and  $\beta$  are embeddings, and by (4) they are inverse of each other. Finally by (1) and (2)

$$d(\alpha \circ \phi, f) < \varpi(\delta_0) + \eta \sum_{n=1}^{+\infty} 2^{-n} < \varpi(\delta).$$

This concludes the proof. The same argument show that there exists a unique separable structure in  $\mathcal{A}$  that satisfies the approximate extension property from Subsection 2.6. A one-sided version of the proof above can be used to prove that any separable structure in  $\mathcal{A}$  embeds into  $M$ .

**2.8. Fraïssé limits: characterization.** It turns out that there are several seemingly different properties that characterize the Fraïssé limit  $M$  up to isomorphism.

**Proposition 2.12.** *Suppose that  $M$  is a separable structure in  $\mathcal{A}$ . The following statements are equivalent.*

- (1)  $M$  is the limit of  $\mathcal{C}$ ;
- (2) For every finitely generated structure  $F$  in  $\mathcal{A}$ , there exists an embedding from  $F$  to  $M$ , and for any  $\delta > 0$ , and morphisms  $\phi : F \rightarrow M$  and  $\psi : F \rightarrow M$  such that  $I(\phi) < \delta$  and  $I(\psi) < \delta$  there exists an automorphism  $\alpha$  of  $M$  such that  $d(\alpha \circ \phi, \psi) < \varpi(\delta)$ ;
- (3) For any finitely generated structures  $E, F$  in  $\mathcal{A}$ ,  $\delta, \varepsilon > 0$ , morphisms  $\phi : E \rightarrow F$  and  $f : E \rightarrow M$  such that  $I(\phi) < \delta$  and  $I(f) < \delta$  there exists a morphism  $g : F \rightarrow M$  such that  $d(g \circ \phi, f) < \varpi(\delta)$  and  $I(g) < \varepsilon$ ;
- (4) For any finitely generated structure  $F$  in  $\mathcal{A}$ , tuple  $\bar{a}$  in  $F$ , embedding  $\phi : \langle \bar{a} \rangle \rightarrow M$ , and  $\varepsilon > 0$ , there exists an embedding  $\psi : F \rightarrow M$  such that  $d(\psi(\bar{a}), \phi(\bar{a})) < \varepsilon$ ;
- (5) For any finitely generated structures  $E$  in  $\mathcal{A}$  and  $F$  in  $\mathcal{I}$ ,  $\varepsilon > 0$ , embeddings  $f : E \rightarrow M$  and  $\phi : E \rightarrow F$  there exists an embedding  $g : F \rightarrow M$  such that  $d(g \circ \phi, f) < \varepsilon$ ;
- (6) Suppose that  $A \in \mathcal{I}$ ,  $\bar{a}$  is a finite tuple in a fixed countable fundamental subset  $D_A$  of  $A$ , and  $f : \langle \bar{a} \rangle \rightarrow M$  is a morphism belonging to a fixed countable uniformly dense collection of morphisms from  $\langle \bar{a} \rangle$  to  $M$ . If  $I(f) \leq \delta$ , then there exists a morphism  $g : A \rightarrow M$  such that  $d(g(\bar{a}), f(\bar{a})) < \varpi(\delta)$  and  $I(g) < \varepsilon$ .

*Proof.* The equivalence of (1) and (4) follows from [7, Corollary 3.20]. The argument of Subsection 2.7 gives a proof of (3) $\Rightarrow$ (2), while the implication (2) $\Rightarrow$ (3) is obvious. Since clearly (2) implies (4), Proposition 2.11 together with uniqueness of the limit shows that (4) and (2) are in fact equivalent. A similar proof as the one in Subsection 2.7 shows that any two separable structures satisfying (5) are isomorphic. This gives the implication (5) $\Rightarrow$ (3), while the converse implication is obvious. The fact that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  and our hypotheses on basic sequences show that (6) implies (4), while the converse implication is obvious.  $\square$

### 3. RETRACTS OF THE LIMIT

**3.1. Approximate injectivity and retracts.** Suppose that  $A$  is an  $\mathcal{L}$ -structure. A *retraction*  $\pi$  of  $A$  is a morphism  $\pi : A \rightarrow A$  that is *idempotent*, that is  $\pi \circ \pi = \pi$ . A *retract* of  $A$  is the image of  $A$  under a retraction. Suppose that  $\mathcal{A}$  is a class of  $\mathcal{L}$ -structures satisfying all the assumptions from Section 2. In the following we will characterize (up to isomorphism) the retracts of the Fraïssé limit  $M$  of the class of finitely generated structures from  $\mathcal{A}$ . The same proof as Lemma 2.10 gives the following lemma.

**Lemma 3.1.** *Suppose that  $X, \hat{X}, Y$  are structures in  $\mathcal{A}$ ,  $\bar{a}$  is a tuple in  $X$ ,  $\phi : X \rightarrow \hat{X}$  and  $f : X \rightarrow Y$  are morphisms such that  $I(\phi) < \delta$ . Then there exists a structure  $\hat{Y}$ , a morphism  $\hat{f} : \hat{X} \rightarrow \hat{Y}$ , and an embedding  $j : Y \rightarrow \hat{Y}$  such that  $d((\hat{f} \circ \phi)(\bar{a}), (j \circ f)(\bar{a})) \leq \varpi(\delta)$  and furthermore for any  $Z \in \mathcal{A}$  and morphisms  $g : \hat{X} \rightarrow Z$  and  $h : \hat{Y} \rightarrow Z$  such that  $d((g \circ \phi)(\bar{a}), (h \circ f)(\bar{a})) \leq \varpi(\delta)$  there exists a morphism  $\tau : \hat{Y} \rightarrow Z$  such that  $g = \tau \circ \hat{f}$  and  $h = \tau \circ j$ . If moreover  $I(f) < \delta$  then  $\hat{f}$  is an embedding. If  $X, \hat{X}, Y$  are finitely generated, then  $\hat{Y}$  is finitely generated.*

The structure  $\hat{Y}$  in Lemma 3.1 together with the canonical morphisms  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  and  $j : Y \rightarrow \hat{Y}$  will be called the *approximate pushout* of  $f$  and  $\phi$  over  $\bar{a}$  with tolerance  $\varpi(\delta)$ . One can similarly define the approximate pushout of a finite sequences of maps  $f_i : X \rightarrow Y_i$  and  $\phi_i : X \rightarrow \hat{X}_i$  over  $\bar{a} \subset X$  with tolerance  $\varpi(\delta_i)$  for  $i = 1, 2, \dots, k$ .

**Definition 3.2.** *Suppose that  $X$  is an  $\mathcal{L}$ -structure in  $\mathcal{A}$ . We say that  $X$  is approximately injective if whenever  $A$  is a structure in  $\mathcal{I}$ ,  $\bar{a}$  is a tuple in  $A$ ,  $f : \langle \bar{a} \rangle \rightarrow X$  is a morphism, and  $\varepsilon > 0$ , there exists a morphism  $g : A \rightarrow X$  such that  $d(g(\bar{a}), f(\bar{a})) \leq \varepsilon$ .*

As observed in Subsection 2.6, the Fraïssé limit  $M$  of the class of finitely generated structures in  $\mathcal{A}$  can be realized as the limit of an inductive sequence of elements of  $\mathcal{I}$  with embeddings as connective maps. It follows from this fact and injectivity of elements of  $\mathcal{I}$  that  $M$  is approximately injective. Therefore any retract of  $M$  is approximately injective as well. The next theorem shows that, conversely, any approximately injective separable structure in  $\mathcal{A}$  is isomorphic as  $\mathcal{L}$ -structure to a retract of  $M$ .

**Theorem 3.3.** *Let  $M$  denote the Fraïssé limit of the class of finitely generated structures in  $\mathcal{A}$ . A separable structure  $X$  in  $\mathcal{A}$  is approximately injective if and only if there exist an embedding  $\phi : X \rightarrow M$  and an idempotent morphism  $\pi : M \rightarrow M$  such that the range of  $\phi$  coincides with the range of  $\pi$ .*

Theorem 3.3 can be proved using the construction of approximate pushouts as in Lemma 3.1. We omit the proof, since we will prove a more general result in Section 7. An alternative proof of Theorem 3.3 follows from the results of Subsection 5.3. Similar characterizations of retracts of Fraïssé limits have been obtained by Dolinka in the countable case [30] and by Kubiś in [66].

**3.2. Approximate injectivity and nuclearity.** We consider now a notion of  $\mathcal{I}$ -nuclearity for structures in  $\mathcal{A}$ ; see Definition 3.4. The term  $\mathcal{I}$ -nuclear is inspired by the characterization of nuclearity for unital  $C^*$ -algebras and operator systems in terms of the completely positive approximation property; see [51] and [15, Section 2.3].

**Definition 3.4.** *A structure  $X$  in  $\mathcal{A}$  is  $\mathcal{I}$ -nuclear if there exist nets  $(\gamma_i)$  and  $(\rho_i)$  of morphisms  $\gamma_i : X \rightarrow A_i$  and  $\rho_i : A_i \rightarrow X$  such that  $A_i \in \mathcal{I}$  and  $\rho_i \circ \gamma_i$  converges pointwise to the identity map of  $X$ .*

We now prove that  $\mathcal{I}$ -nuclearity is equivalent to approximate injectivity. If  $f, g : E \rightarrow F$  are functions between  $\mathcal{L}$ -structures, and  $\bar{a}$  is an  $n$ -tuple in  $E$ , we write  $f \approx_{\bar{a}, \varepsilon} g$  to express the fact that  $d(f(\bar{a}), g(\bar{a})) \leq \varepsilon$ .

**Proposition 3.5.** *Suppose that  $X$  is a structure in  $\mathcal{A}$ . The following assertions are equivalent:*

- (1)  $X$  is approximately injective;
- (2)  $X$  is  $\mathcal{I}$ -nuclear;
- (3) *Whenever  $E, F$  are finitely generated structures in  $\mathcal{A}$ ,  $\bar{a}$  is a finite tuple in  $E$ ,  $\phi : E \rightarrow F$  and  $f : E \rightarrow X$  are morphisms such that  $I(\phi) < \delta$ , there exists a morphism  $g : F \rightarrow X$  such that  $g \circ \phi \approx_{\bar{a}, \varpi(\delta)} f$ .*

*Proof.* We present the proofs of the nontrivial implications below.

(1) $\Rightarrow$ (2): If  $X$  is approximately injective, then by Theorem 3.3  $X$  is isomorphic to a retract of the Fraïssé limit  $M$  of the class of finite-dimensional structures in  $\mathcal{A}$ . Therefore it is enough to prove that  $M$  is  $\mathcal{I}$ -nuclear. Recall that  $M$  contains an increasing sequence  $(B_n)$  of structures from  $\mathcal{I}$  with dense union. Therefore it is enough to prove that if  $\bar{a} \subset B_n \subset M$  is a finite tuple and  $\varepsilon > 0$ , then there exist morphisms  $\gamma : M \rightarrow B_n$  and  $\rho : B_n \rightarrow M$  such that  $(\rho \circ \gamma)(\bar{a}) = \bar{a}$ . Consider the identity map of  $B_n$  and observe that by injectivity of  $B_n$  it extends to a morphism  $\gamma : M \rightarrow B_n$ . Let now  $\rho : B_n \rightarrow M$  be the inclusion map and observe that  $(\gamma \circ \rho)(\bar{a}) = \bar{a}$ .

(2) $\Rightarrow$ (3): Let  $E, F, \bar{a}, \phi, f$  be as in (3). Let  $\delta_0 > 0$  be such that  $I(\phi) < \delta_0 < \delta$ . Fix also  $\varepsilon > 0$  such that  $\varpi(\delta_0) + \varepsilon < \varpi(\delta)$ . By assumption there exist  $A \in \mathcal{I}$  and morphisms  $\gamma : X \rightarrow A$  and  $\rho : A \rightarrow X$  such that  $\rho \circ \gamma \circ f \approx_{\bar{a}, \varepsilon} f$ . Since  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  with modulus  $\varpi$  there exists a morphism  $h : F \rightarrow A$  such that  $d(h \circ \phi, f) \leq \varpi(\delta_0)$ . Set  $g = \rho \circ h$  and observe that  $d((g \circ \phi)(\bar{a}), f(\bar{a})) \leq \varpi(\delta_0) + \varepsilon < \varpi(\delta)$ .  $\square$

**3.3.  $\mathcal{I}$ -structures.** In the following if  $A, B$  are subsets of a structure  $X$ , we write  $A \subseteq_\varepsilon B$  if every element of  $A$  is at distance at most  $\varepsilon$  from some element of  $B$ .

**Definition 3.6.** *We say that structure  $X$  in  $\mathcal{A}$  is an  $\mathcal{I}$ -structure if for every finitely generated substructure  $E$  of  $X$  and  $\varepsilon > 0$  there exist a finitely generated substructure  $B$  of  $X$  containing  $E$ , a structure  $\hat{B}$  in  $\mathcal{I}$  such that  $d(B, \hat{B}) < \varepsilon$ . We say that  $X$  is a rigid  $\mathcal{I}$ -structure if for every finite subset  $x_1, \dots, x_n$  of  $X$  there exists a substructure  $A$  of  $X$  that belongs to  $\mathcal{I}$  such that  $\{x_1, \dots, x_n\} \subseteq_\varepsilon A$ .*

Every structure in  $\mathcal{A}$  that can be represented as the direct limit of elements of  $\mathcal{I}$  with embeddings as connective maps is clearly a rigid  $\mathcal{I}$ -structure. Particularly, the Fraïssé limit  $M$  of finite-dimensional structures in  $\mathcal{A}$  is a rigid  $\mathcal{I}$ -structure. In turn, it follows from injectivity of elements of  $\mathcal{I}$  together with the fact that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  and our assumptions on basic sequences that any rigid  $\mathcal{I}$ -structure is an  $\mathcal{I}$ -structure, and that an  $\mathcal{I}$ -structure is approximately injective. The following proposition provides a characterization among the (rigid)  $\mathcal{I}$ -structures of the Fraïssé limit of the class of finitely generated structures in  $\mathcal{A}$ .

**Proposition 3.7.** *Let  $X$  be a separable structure in  $\mathcal{A}$ . The following statements are equivalent:*

- (1)  $X$  is the Fraïssé limit  $M$  of the class of finitely generated structures in  $\mathcal{A}$ ;
- (2)  $M$  is an  $\mathcal{I}$ -structure, and for any  $\delta, \varepsilon > 0$ , structures  $A, \hat{A} \in \mathcal{I}$ , embedding  $\phi : A \rightarrow \hat{A}$ , and morphism  $f : A \rightarrow X$  such that  $I(f) < \delta$ , there exists a morphism  $\hat{f} : \hat{A} \rightarrow X$  such that  $I(\hat{f}) < \varepsilon$  and  $d(\hat{f} \circ \phi, f) < \varpi(\delta)$ ;
- (3)  $M$  is a rigid  $\mathcal{I}$ -structure, and for any structures  $A, \hat{A} \in \mathcal{I}$ , embeddings  $\phi : A \rightarrow \hat{A}$  and  $f : A \rightarrow X$ , and  $\varepsilon > 0$ , there exists a morphism  $\hat{f} : \hat{A} \rightarrow X$  such that  $I(\hat{f}) < \varepsilon$  and  $d(\hat{f} \circ \phi, f) < \varepsilon$ .

*Proof.* The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from Proposition 2.12 and the already observed fact that the Fraïssé limit of the class of finitely generated structures in  $\mathcal{A}$  is a rigid  $\mathcal{I}$ -structure.

We now prove that (2) implies (1). Fix a countable fundamental subset  $D_X$  of  $X$  as in Definition 2.5 and a sequence  $(\delta_n)$  of strictly positive real numbers such that  $\sum_n \varpi(2\delta_n) < +\infty$ . Using the hypothesis, and proceeding as in Subsection 2.6, one can define by recursion on  $n$ :

- structures  $B_n, \tilde{C}_n \in \mathcal{I}$ , and substructures  $C_n$  of  $X$ ,
- morphisms  $\alpha_n : B_n \rightarrow C_n$ ,  $f_n : C_n \rightarrow \tilde{C}_n$ ,  $g_n : \tilde{C}_n \rightarrow C_n$ , and embeddings  $\beta_n : C_n \rightarrow B_{n+1}$  and  $\phi_n : B_n \rightarrow B_{n+1}$ ,

such that

- $\{x_1, \dots, x_n\} \subset_{\delta_n} C_n$ ,
- $I(\alpha_n) < \delta_n$ ,  $I(f_n) < \delta_n$ ,  $I(g_n) < \delta_n$ ,
- $d(f_n \circ g_n, id_{\tilde{C}_n}) < \delta_n$ ,  $d(g_n \circ f_n, id_{C_n}) < \delta_n$ ,  $d(\beta_n \circ \alpha_n, \phi_n) < \varpi(\delta_n)$ , and  $d(\alpha_{n+1} \circ \beta_n, g_n) < \varpi(2\delta_n)$ , where  $\iota_n : C_n \rightarrow X$  is the inclusion map, and
- the limit of the inductive sequence  $(B_n)$  with connective maps  $\phi_n$  is the Fraïssé limit  $M$  of the class of finitely generated structures in  $\mathcal{A}$ .

Suppose that we have defined  $B_k, \alpha_k, C_k, \tilde{C}_k, f_k, g_k, \phi_{k-1}, \beta_{k-1}$  for  $k \leq n$ . Proceeding as in Subsection 2.6 one can define a structure  $B_{n+1} \in \mathcal{I}$  and an embedding  $\phi_n : B_n \rightarrow B_{n+1}$  satisfying all the requirements of the  $n$ -th step of Subsection 2.6. Using the recursion hypothesis, we can moreover guarantee that there exists a morphism  $\beta_n : \tilde{C}_n \rightarrow B_{n+1}$  such that  $d(\beta_n \circ f_n \circ \alpha_n, \phi_n) < \varpi(\delta_n)$ . We apply now the hypothesis to  $g_n \circ \beta_n^{-1}$  to define a morphism  $\alpha_{n+1} : B_{n+1} \rightarrow X$  such that  $I(\alpha_{n+1}) < \delta_n$  and  $d(\alpha_{n+1} \circ \beta_n, g_n) < \varpi(2\delta_n)$ . Define  $C_{n+1}$  to be the range of  $\beta_{n+1}$ . Finally one can obtain  $\tilde{C}_{n+1}$ ,  $f_{n+1}$ , and  $g_{n+1}$  by applying the hypothesis that  $X$  is an  $\mathcal{I}$ -structure. This concludes the recursive construction. Granted the construction, the sequences of morphisms  $(\alpha_k)$  induces at the limit a morphisms  $\alpha : M \rightarrow X$ . Such a morphism is well defined by (c), it is an embedding by (b), and it is onto by (a) and (c).

We now prove that (3) implies (1). Fix a dense sequence  $(x_n)$  of elements of  $X$  and a sequence  $(\delta_n)$  of strictly positive real numbers that converges to 0 fast enough. One can define by recursion on  $n$ :

- structures  $B_n, C_n \in \mathcal{I}$  with  $C_n \subset X$ ,
- morphisms  $\alpha_n : B_n \rightarrow C_n$  and embeddings  $\beta_n : C_n \rightarrow B_{n+1}$  and  $j_n : B_n \rightarrow B_{n+1}$ ,

such that, if  $\iota_n : C_n \rightarrow X$  is the inclusion map, then

- $\{x_1, \dots, x_n\} \subset_{\delta_n} C_n$ ,
- $I(\alpha_n) < \delta_n$ ,
- $d(i_{n+1} \circ \alpha_n, i_n) < \delta_n$ , and  $d(\beta_{n+1} \circ \alpha_n, j_n) < \varpi(\delta_n)$ ,
- the limit of the inductive sequence  $(B_n)$  with connective maps  $j_n : B_n \rightarrow B_{n+1}$  is isomorphic to the Fraïssé limit of the class of finitely generated structures in  $\mathcal{A}$ .

This can be seen proceeding as the proof of (2) $\Rightarrow$ (1), using furthermore the assumption that  $X$  is a rigid  $\mathcal{I}$ -structure and the construction of the approximate pushout from Lemma 2.9.  $\square$

#### 4. UNIVERSAL MORPHISMS

Throughout this section and the next section we will use the same notation and terminology as in Section 2. Particularly we will suppose that  $\mathcal{L}$  is a language in the logic for metric structures,  $\mathcal{A}$  is a class of  $\mathcal{L}$ -structures, and  $\mathcal{I} \subset \mathcal{A}$  is a countable collection of finitely generated injective structures satisfying the assumptions of Theorem 2.8 such that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  with modulus  $\varpi$ . Again we stick for simplicity to the case when  $\mathcal{L}$  is single-sorted.

**4.1. Rota universal operators.** In [100, 101] Rota constructed a surjective contractive linear operator  $\Omega$  on  $\ell^2$  which is a universal model for bounded linear operators on separable Hilbert spaces. This means that if  $H_0, H_1$  are separable Hilbert spaces and  $T : H_0 \rightarrow H_1$  is a bounded linear operator, then there exist injective bounded linear maps  $\alpha_0 : H_0 \rightarrow \ell^2$  and  $\alpha_1 : H_1 \rightarrow \ell^2$  such that  $\alpha_1 \circ T = \Omega \circ \alpha_0$ . Clearly, it follows that when  $H_0 = H_1$  one can take  $\alpha_0 = \alpha_1$ . An example of such an operator is the infinite amplification on the unilateral shift on  $\ell^2$ . Rota's original motivation comes from the invariant subspace problem for operators on the separable infinite-dimensional Hilbert space. Operators that are universal in the sense of Rota have been characterized in [17], and are currently the subject of active research; see for example [21, 22].

An analogue of Rota's universal operator for the class of operators on arbitrary separable Banach spaces was constructed by Garbulińska-Węgrzyn and Kubiś in [45]. In this section we will prove a general result concerning the existence of a "universal morphism" defined on the Fraïssé limit  $M$  of a Fraïssé class  $\mathcal{C}$  as in Theorem 2.8. As a consequence of our general results from this and the next section, we can give an explicit characterization of

the universal operator constructed by Garbulińska-Węgrzyn and Kubiś; see Theorem 4.1 below and Subsection 6.1.

A *quotient mapping*  $\phi : X \rightarrow Y$  between Banach spaces is a linear function that sends the open unit ball of  $X$  onto the open unit ball of  $Y$ . This is equivalent to the assertion that the map  $X/\text{Ker}(\phi) \rightarrow Y$  induced by  $\phi$  is a surjective linear isometry. Recall that the Lusky simplex  $\mathbb{L}$  is the unit ball of the dual space of the Gurarij space  $\mathbb{G}$ . The definition of  $M$ -ideal in a Banach space can be found in Subsection 6.1; see also [2, 3].

**Theorem 4.1.** *Suppose that  $T : \mathbb{G} \rightarrow \mathbb{G}$  is a linear map of norm at most 1,  $N$  is the kernel of  $T$ , and  $H = N^\perp \cap \mathbb{L}$ . The following assertions are equivalent:*

- (1)  *$T$  is a quotient mapping and  $N$  is a nonzero  $M$ -ideal of  $\mathbb{G}$ ;*
- (2)  *$T$  is a quotient mapping and  $H$  is a closed proper biface of  $\mathbb{L}$  symmetrically affinely homeomorphic to  $\mathbb{L}$ ;*
- (3) *whenever  $E_0 \subset F_0$  and  $E_1 \subset F_1$  are finite-dimensional Banach spaces,  $f_0 : E_0 \rightarrow \mathbb{G}$  and  $f_1 : E_1 \rightarrow \mathbb{G}$  are linear isometries,  $L : F_0 \rightarrow F_1$  is a linear map of norm at most 1 mapping  $E_0$  to  $E_1$  such that  $T \circ f_0 = f_1 \circ L$ , and  $\varepsilon > 0$ , then there exist linear isometries  $\hat{f}_0 : F_0 \rightarrow \mathbb{G}$  and  $\hat{f}_1 : F_1 \rightarrow \mathbb{G}$  such that  $\|T \circ \hat{f}_0 - \hat{f}_1 \circ L\| < \varepsilon$ .*

*The set of operators satisfying the equivalent conditions above is a dense  $G_\delta$  subset of the space  $\text{Ball}(B(\mathbb{G}))$  of linear operators on  $\mathbb{G}$  of norm at most 1, and forms a single orbit under the action  $\text{Aut}(\mathbb{G}) \curvearrowright \text{Ball}(B(\mathbb{G}))$ ,  $(\alpha, S) \mapsto S \circ \alpha^{-1}$ . If  $\Omega_{\mathbb{G}} : \mathbb{G} \rightarrow \mathbb{G}$  is such an operator, then the kernel of  $\Omega_{\mathbb{G}}$  is isometrically isomorphic to  $\mathbb{G}$ . In other words the sequence*

$$0 \longrightarrow \mathbb{G} \longrightarrow \mathbb{G} \xrightarrow{\Omega_{\mathbb{G}}} \mathbb{G} \longrightarrow 0$$

*where the first arrow is a linear isometry, is exact. Furthermore  $\Omega_{\mathbb{G}}$  is a universal operator between separable Banach spaces, in the sense that any if  $L : E_0 \rightarrow E_1$  is a linear map of norm at most 1 between separable Banach spaces, then there exist linear isometries  $\eta_0 : E_0 \rightarrow \mathbb{G}$  and  $\eta_1 : E_1 \rightarrow \mathbb{G}$  such that  $\Omega_{\mathbb{G}} \circ \eta_0 = \eta_1 \circ L$ .*

A similar result holds for complex scalars; see 6.2. We will also prove in Subsection 6.3 the analogous statement for the space of affine functions on the Poulsen simplex.

**Theorem 4.2.** *Suppose that  $T : A(\mathbb{P}) \rightarrow A(\mathbb{P})$  is a unital positive linear map,  $N$  is the kernel of  $T$ , and  $H = N^\perp \cap \mathbb{P}$ . The following assertions are equivalent:*

- (1)  *$T$  is a quotient mapping and  $N$  is a nonzero  $M$ -ideal of  $A(\mathbb{P})$ ;*
- (2)  *$T$  is a quotient mapping and  $H$  is a closed proper face of  $\mathbb{P}$ ;*
- (3) *whenever  $E_0 \subset F_0$  and  $E_1 \subset F_1$  are finite-dimensional function systems,  $f_0 : E_0 \rightarrow A(\mathbb{P})$  and  $f_1 : E_1 \rightarrow A(\mathbb{P})$  are unital linear isometries,  $L : F_0 \rightarrow F_1$  is a unital positive linear function mapping  $E_0$  to  $E_1$  such that  $T \circ f_0 = f_1 \circ L$ , and  $\varepsilon > 0$ , then there exist unital linear isometries  $\hat{f}_0 : F_0 \rightarrow A(\mathbb{P})$  and  $\hat{f}_1 : F_1 \rightarrow A(\mathbb{P})$  such that  $\|T \circ \hat{f}_0 - \hat{f}_1 \circ L\| < \varepsilon$ .*

*The set of unital positive linear maps satisfying the equivalent conditions above is a dense  $G_\delta$  subset of the space  $\text{UP}(A(\mathbb{P}))$  of unital positive linear maps on  $A(\mathbb{P})$ , and forms a single orbit under the action  $\text{Aut}(A(\mathbb{P})) \curvearrowright \text{UP}(A(\mathbb{P}))$ ,  $(\alpha, S) \mapsto S \circ \alpha^{-1}$ . If  $\Omega_{A(\mathbb{P})} : A(\mathbb{P}) \rightarrow A(\mathbb{P})$  is such an operator, then the set*

$$\{x \in A(\mathbb{P}) : \Omega_{A(\mathbb{P})}(x) \text{ is a scalar multiple of the identity}\}$$

*is a function system unitaly isometrically isomorphic to  $A(\mathbb{P})$ .*

As a further application of the general results of this section we will obtain the existence of a noncommutative analog of the Garbulińska-Węgrzyn-Kubiś operator defined on the noncommutative Gurarij space [91, 81]; see Subsection 8.1.

**Theorem 4.3.** *There exists a complete quotient mapping  $\Omega_{\text{NG}} : \text{NG} \rightarrow \text{NG}$  such that, if  $T : X \rightarrow Y$  is a completely contractive linear map between separable exact operator spaces, then there exist completely isometric linear maps  $\alpha_0 : X \rightarrow \text{NG}$  and  $\alpha_1 : Y \rightarrow \text{NG}$  such that  $\alpha_1 \circ T = \Omega_{\text{NG}} \circ \alpha_0$ . Furthermore  $\Omega_{\text{NG}}$  is generic in the sense that the orbit  $\{\Omega_{\text{NG}} \circ \beta : \beta \in \text{Aut}(\text{NG})\}$  with respect to the continuous action  $\text{Aut}(\text{NG}) \curvearrowright \text{Ball}(B(\text{NG}))$ ,  $(\alpha, T) \mapsto T \circ \alpha^{-1}$  is a dense  $G_\delta$  subspace of the space  $\text{Ball}(B(\text{NG}))$  of linear complete contractions on  $\text{NG}$  endowed with the topology of pointwise convergence. The kernel of  $\Omega_{\text{NG}}$  is completely isometric to  $\text{NG}$ . In other words there exists an exact sequence*

$$0 \longrightarrow \text{NG} \longrightarrow \text{NG} \xrightarrow{\Omega_{\text{NG}}} \text{NG} \longrightarrow 0$$

where the second arrow is a linear complete isometry. A completely contractive linear map  $T : \mathbb{NG} \rightarrow \mathbb{NG}$  belongs to the  $\text{Aut}(\mathbb{NG})$ -orbit of  $\Omega_{\mathbb{NG}}$  if and only if it satisfies the following property: whenever  $E_0 \subset F_0$  and  $E_1 \subset F_1$  are finite-dimensional exact operator spaces,  $f_0 : E_0 \rightarrow \mathbb{NG}$  and  $f_1 : E_1 \rightarrow \mathbb{NG}$  are linear complete isometries,  $L : F_0 \rightarrow F_1$  is a linear complete contraction mapping  $E_0$  to  $E_1$  such that  $T \circ f_0 = f_1 \circ L$ , and  $\varepsilon > 0$ , then there exist linear complete isometries  $\hat{f}_0 : F_0 \rightarrow \mathbb{NG}$  and  $\hat{f}_1 : F_1 \rightarrow \mathbb{NG}$  such that  $\|T \circ \hat{f}_0 - \hat{f}_1 \circ L\|_{cb} < \varepsilon$ .

The same result holds in the operator systems category, yielding a universal unital completely positive map  $\Omega_{A(\mathbb{NP})}$  defined on the noncommutative Poulsen system  $A(\mathbb{NP})$ ; see Subsection 8.2.

**Theorem 4.4.** *There exists a unital completely positive quotient mapping  $\Omega_{A(\mathbb{NP})} : A(\mathbb{NP}) \rightarrow A(\mathbb{NP})$  such that, if  $T : X \rightarrow Y$  is a unital completely positive linear map between separable exact operator systems, then there exist unital completely isometric linear maps  $\alpha_0 : X \rightarrow \mathbb{NG}$  and  $\alpha_1 : Y \rightarrow \mathbb{NG}$  such that  $\alpha_1 \circ T = \Omega_{\mathbb{NG}} \circ \alpha_0$ . Furthermore  $\Omega_{A(\mathbb{NP})}$  is generic in the sense that the orbit  $\{\Omega_{A(\mathbb{NP})} \circ \beta : \beta \in \text{Aut}(A(\mathbb{NP}))\}$  with respect to the continuous action  $\text{Aut}(A(\mathbb{NP})) \curvearrowright \text{UCP}(A(\mathbb{NP}))$  is a dense  $G_\delta$  subspace of the space  $\text{UCP}(A(\mathbb{NP}))$  of unital completely positive maps from  $A(\mathbb{NP})$  to itself endowed with the topology of pointwise convergence. The set*

$$\{x \in A(\mathbb{NP}) : \Omega_{A(\mathbb{NP})}(x) \text{ is a scalar multiple of the identity}\}$$

*is unittally completely isometrically isomorphic to  $A(\mathbb{NP})$ . A unital completely positive map  $T : A(\mathbb{NP}) \rightarrow A(\mathbb{NP})$  belongs to the  $\text{Aut}(A(\mathbb{NP}))$ -orbit of  $\Omega_{\mathbb{NG}}$  if and only if it satisfies the following property: whenever  $E_0 \subset F_0$  and  $E_1 \subset F_1$  are finite-dimensional exact operator spaces,  $f_0 : E_0 \rightarrow A(\mathbb{NP})$  and  $f_1 : E_1 \rightarrow A(\mathbb{NP})$  are unital linear complete isometries,  $L : F_0 \rightarrow F_1$  is a unital completely positive linear function mapping  $E_0$  to  $E_1$  such that  $T \circ f_0 = f_1 \circ L$ , and  $\varepsilon > 0$ , then there exist unital linear complete isometries  $\hat{f}_0 : F_0 \rightarrow A(\mathbb{NP})$  and  $\hat{f}_1 : F_1 \rightarrow A(\mathbb{NP})$  such that  $\|T \circ \hat{f}_0 - \hat{f}_1 \circ L\|_{cb} < \varepsilon$ .*

Results analogous to Theorem 4.1, Theorem 4.2, Theorem 4.3, and Theorem 4.4 also hold for  $M_q$ -spaces,  $M_q$ -systems, operator sequence spaces, and  $p$ -multinormed spaces; see Subsections 6.6, 6.7, 6.5, and 6.4.

**4.2. Morphisms between morphisms.** We can regard morphisms between structures in  $\mathcal{A}$  as objects of a category  $\mathcal{A}^\rightarrow$ . Suppose that  $T : X \rightarrow Y$  is a morphism between structures in  $\mathcal{A}$ . We use the notation  $D_0(T)$  and  $D_1(T)$  to denote the domain and the codomain of  $T$ , respectively. A morphism in  $\mathcal{A}^\rightarrow$  from the morphism  $T : D_0(T) \rightarrow D_1(T)$  to the morphism  $S : D_0(S) \rightarrow D_1(S)$  is given by a pair  $\alpha = (\alpha_0, \alpha_1)$ , where  $\alpha_0 : D_0(T) \rightarrow D_0(S)$  and  $\alpha_1 : D_1(T) \rightarrow D_1(S)$  are morphisms in  $\mathcal{A}$ . We do not require that  $\alpha_1 \circ T = S \circ \alpha_0$ . If  $\alpha$  is a morphism from  $T$  to  $S$  as above, then we set  $I(\alpha)$  to be maximum of  $I(\alpha_0)$ ,  $I(\alpha_1)$ , and

$$\sup_x d((\alpha_1 \circ T)(x), (S \circ \alpha_0)(x))$$

where  $x$  ranges in  $D_0(X)$ , and  $I(\alpha_0)$ ,  $I(\alpha_1)$  are defined as in Subsection 2.1. Observe that  $I(\alpha)$  measures how close  $\alpha$  is to be a pair of embeddings that commute with  $T$  and  $S$ . If  $\alpha, \beta$  are morphisms from  $T$  to  $S$  then we set  $d(\alpha, \beta)$  to be the maximum of  $d(\alpha_0, \beta_0)$  and  $d(\alpha_1, \beta_1)$ . An *embedding* from  $T$  to  $S$  is a morphism  $\alpha$  as above such that moreover  $\alpha_0, \alpha_1$  are isometries and  $\alpha_1 \circ T = S \circ \alpha_0$ . An *automorphism* of  $T$  is an embedding  $(\alpha_0, \alpha_1)$  from  $T$  to  $T$  such that  $\alpha_0$  and  $\alpha_1$  are surjective.

Observe that the objects of  $\mathcal{A}^\rightarrow$  can naturally be regarded as structures in a language  $\mathcal{L}^\rightarrow$ . Here  $\mathcal{L}^\rightarrow$  is the two-sorted language in sorts  $D_0$  and  $D_1$  that has

- an  $n$ -ary function symbols  $f_i : D_i^n \rightarrow D_i$  for every  $i \in \{0, 1\}$  and every  $n$ -ary function symbol  $f$  in  $\mathcal{L}$ ,
- an  $n$ -relation symbol  $R_i : D_i^n \rightarrow [0, 1]$  for every  $i \in \{0, 1\}$  and every  $n$ -ary relation symbol  $R$  in  $\mathcal{L}$ ,
- a unary function symbol  $D_0 \rightarrow D_1$ .

Clearly a structure  $T$  in  $\mathcal{A}^\rightarrow$  is finitely generated as  $\mathcal{L}^\rightarrow$ -structure if and only if both  $D_0(T)$  and  $D_1(T)$  are finitely generated as  $\mathcal{L}$ -structures.

**4.3. The generic morphism.** Let  $\mathcal{C}^\rightarrow \subset \mathcal{A}^\rightarrow$  be the class of morphisms between finitely generated structures in  $\mathcal{A}$ . We aim at showing that  $\mathcal{C}^\rightarrow$  is a (complete) Fraïssé class in the sense of [7, Definition 3.15]. The fact that the class  $\mathcal{C}_n^\rightarrow$  of  $n$ -marked structures in  $\mathcal{A}^\rightarrow$  is complete and separable can be proved as in Subsection 2.5. The same holds for the hereditary property and the joint embedding property. It remains to prove the near amalgamation property.

**Lemma 4.5.** *Suppose that  $T, \hat{T}, S$  are structures in  $\mathcal{A}^\rightarrow$ ,  $\phi : T \rightarrow \hat{T}$  and  $f : T \rightarrow S$  are morphisms such that  $I(\phi) \leq \delta$ . Then there exist a structure  $\hat{S}$  in  $\mathcal{A}^\rightarrow$ , a morphism  $\hat{f} : \hat{T} \rightarrow \hat{S}$ , and an embedding  $j : S \rightarrow \hat{S}$  such that  $\hat{S} \circ \hat{f}_0 = \hat{f}_1 \circ \hat{T}$  and  $d(\hat{f} \circ \phi, j \circ f) \leq \varpi(\delta) + 2\delta$ . If moreover  $I(f) \leq \delta$  then  $\hat{f}$  is an embedding. If  $T, \hat{T}, S$  are finitely generated, then  $\hat{S}$  is finitely generated.*

*Proof.* Let  $D_1(\widehat{S})$  be the approximate pushout of  $f_1$  and  $\phi_1$  with tolerance  $\varpi(\delta)$  defined as in Lemma 2.10. Consider also the canonical embedding  $j_1 : D_1(S) \rightarrow D_1(\widehat{S})$  and the canonical morphism  $\widehat{f}_1 : D_1(\widehat{T}) \rightarrow D_1(\widehat{S})$ . Define  $D_0(\widehat{S})$  to be the approximate pushout of  $f_0$  and  $\phi_0$  with tolerance  $\varpi(\delta) + 2\delta$ . Again we have a canonical embedding  $j_0 : D_0(S) \rightarrow D_0(\widehat{S})$  and a canonical morphism  $\widehat{f}_0 : D_0(\widehat{T}) \rightarrow D_0(\widehat{S})$ . Observe now that  $j_1 \circ S : D_0(S) \rightarrow D_1(\widehat{S})$  and  $\widehat{f}_1 \circ \widehat{T} : D_0(\widehat{T}) \rightarrow D_1(\widehat{S})$  are morphisms such that

$$\begin{aligned} d(j_1 \circ S \circ f_0, \widehat{f}_1 \circ \widehat{T} \circ \phi_0) &\leq d(S \circ f_0, f_1 \circ T) + d(\widehat{T} \circ \phi_0, \phi_1 \circ T) + d(j_1 \circ f_1, \widehat{f}_1 \circ \phi_1) \\ &\leq \varpi(\delta) + 2\delta. \end{aligned}$$

Therefore by the universal property of the approximate pushout there exists a unique morphism  $\widehat{S} : D_0(\widehat{S}) \rightarrow D_1(\widehat{S})$  such that  $\widehat{S} \circ \widehat{f}_0 = \widehat{f}_1 \circ \widehat{T}$  and  $\widehat{S} \circ j_0 = j_1 \circ S$ . The same construction also works to prove the other assertions.  $\square$

One can also consider in this context an analog of Lemma 3.1 involving approximate pushouts over a tuple. It is immediate to observe that Lemma 4.5 shows that  $\mathcal{C}^\rightarrow$  has the near amalgamation property. We can therefore conclude that  $\mathcal{C}^\rightarrow$  is a Fraïssé class.

**Proposition 4.6.** *The class  $\mathcal{C}^\rightarrow$  of finitely generated  $\mathcal{L}^\rightarrow$ -structures in  $\mathcal{A}^\rightarrow$  is a Fraïssé class. The corresponding Fraïssé limit is a morphism  $\Omega_M : M \rightarrow M$ , where  $M$  is the Fraïssé limit of the class  $\mathcal{C}$  of finitely generated  $\mathcal{L}$ -structures in  $\mathcal{A}$ .*

*Proof.* We have shown above that the collection  $\mathcal{C}^\rightarrow$  of finitely generated structures in  $\mathcal{A}^\rightarrow$  is a Fraïssé class. The corresponding limit is a morphism  $\Omega_M : D_0(\Omega_M) \rightarrow D_1(\Omega_M)$ . Using the characterization of the Fraïssé limit from Proposition 2.12—see also [7, Corollary 3.20]—one can conclude that  $D_0(\Omega_M)$  and  $D_1(\Omega_M)$  satisfy the characterizing property of the Fraïssé limit  $M$  of the class  $\mathcal{C}$  of finitely generated structures in  $\mathcal{A}$ . Therefore  $D_0(\Omega_M)$  and  $D_1(\Omega_M)$  are both isomorphic to  $M$ .  $\square$

It follows from universality of the Fraïssé limit that  $\Omega_M$  is a *universal morphism* between separable structures in  $\mathcal{A}$ . This means that if  $T : D_0(T) \rightarrow D_1(T)$  is a morphism between separable structures in  $\mathcal{A}$ , then there exist embeddings  $\phi_0 : D_0(T) \rightarrow M$  and  $\phi_1 : D_1(T) \rightarrow M$  such that  $\Omega_M \circ \phi_0 = \phi_1 \circ T$ .

One can prove a characterization of  $\Omega_M$  similar to the characterization of  $M$  given by Proposition 2.12. In particular if  $S : M \rightarrow M$  is a morphism, then the following statements are equivalent:

- (1) There exists an automorphism  $\alpha_0, \alpha_1$  of  $M$  such that  $\alpha_0 \circ \Omega_M \circ \alpha_1 = S$ ;
- (2) For every morphisms  $T, \widehat{T}$  between finitely generated structures in  $\mathcal{A}$ ,  $\delta > 0$ , morphisms  $f : T \rightarrow S$  and  $\phi : T \rightarrow \widehat{T}$  such that  $I(f) < \delta$  and  $I(\phi) < \delta$ , there exists an embedding  $g : \widehat{T} \rightarrow S$  such that  $d(g \circ \phi, f) < \varpi(\delta) + 2\delta$ ;
- (3) For every morphisms  $T, \widehat{T}$  between finitely generated structures in  $\mathcal{A}$ , embeddings  $f : T \rightarrow S$  and  $\phi : T \rightarrow \widehat{T}$ , and  $\varepsilon > 0$ , there exists an embedding  $g : \widehat{T} \rightarrow S$  such that  $d(g \circ \phi, f) < \varepsilon$ ;
- (4) Whenever  $T$  is a morphism between finitely-generated structures in  $\mathcal{A}$ ,  $f : T \rightarrow S$  and  $\phi : T \rightarrow S$  are morphisms such that  $I(f) < \delta$  and  $I(\phi) < \delta$ , there exists an automorphism  $\beta$  of  $M$  such that  $\beta \circ S = S \circ \beta$  and  $d(\beta \circ \phi, f) < \varpi(\delta) + 2\delta$ ;
- (5) For any finite tuple  $\bar{a}$  in  $M$ , morphisms  $f : T|_{\langle \bar{a} \rangle} \rightarrow S$  and  $\phi : T|_{\langle \bar{a} \rangle} \rightarrow S$  such that  $I(f) < \delta$ ,  $I(\phi) < \delta$ , there exists an automorphism  $\beta$  of  $M$  such that  $S \circ \beta = \beta \circ S$  and  $d(\beta \circ \phi, f) < \varpi(\delta) + 2\delta$ ;
- (6) The same as (5) where the tuple  $\bar{a}$  belongs to some fixed countable fundamental subset of  $M$ , and  $\phi(\bar{a}), f(\bar{a})$  belong to some fixed countable fundamental subset of  $M$ .

One can deduce from such a characterization that the orbit  $\{\alpha \circ \Omega_M \circ \alpha_1 : \alpha_0, \alpha_1 \in \text{Aut}(M)\}$  of  $\Omega_M$  is a dense  $G_\delta$  subset of  $\text{End}(M)$ . Here  $\text{End}(M)$  is the Polish space of morphisms  $S : M \rightarrow M$  endowed with the topology of pointwise convergence, and  $\text{Aut}(M) \subset \text{End}(M)$  is the  $G_\delta$  subspace of automorphisms of  $M$ .

Recall our assumption from Subsection 2.2 that  $\mathcal{A}$  has a universal initial object  $A_0$  that is a finitely-generated structure.

**Proposition 4.7.** *Suppose that  $A_0$  is also a universal initial object in the category that has the same objects as  $\mathcal{A}$  and embeddings as morphisms. Identify canonically  $A_0$  with a substructure of any object of  $\mathcal{A}$ . Assume furthermore that for any structure  $X$  in  $\mathcal{A}$  there exists a morphism from  $X$  to  $A_0$ . If  $f : X \rightarrow Y$  is a morphism, we set  $\text{Ker}(f) = \{x \in X : f(x) \in A_0\}$ . Then the morphism  $\Omega_M$  is surjective and  $\text{Ker}(\Omega_M)$  is isomorphic to  $M$ .*

*Proof.* In order to prove that  $\Omega_M$  is surjective, it is enough to show that the range of  $\Omega_M$  is dense. Fix  $y \in M$  and  $\varepsilon > 0$ . Let  $\langle y \rangle$  be the substructure of  $M$  generated by  $y$ . Observe that  $A_0 \subset \langle y \rangle$ . By the characterization of  $\Omega_M$ , there exist embeddings  $\psi_0, \psi_1 : \langle y \rangle \rightarrow M$  such that  $d(\psi_1(y), y) < \varepsilon$  and  $\Omega_M \circ \psi_0 = \psi_1$ . Therefore



$\psi_1(y) = \Omega_M(x)$  where  $x = \psi_0(y)$  and  $d(y, \Omega_M(x)) < \varepsilon$ . This concludes the proof that the range of  $\Omega_M$  is dense. We now show that  $\text{Ker}(\Omega_M)$  is isomorphic to  $M$ . Suppose that  $E$  is a finitely generated structure in  $\mathcal{A}$ ,  $\bar{a}$  is a finite tuple in  $E$ , and  $\phi : \langle \bar{a} \rangle \rightarrow \text{Ker}(\Omega_M)$  is an embedding. Let  $T : E \rightarrow A_0$  be a morphism. By the properties of  $\Omega_M$  there exists an embedding  $\psi : E \rightarrow M$  such that  $d(\psi(\bar{a}), \phi(\bar{a})) < \varepsilon$  and  $\Omega_M \circ \psi = T$ . This implies that the range of  $\psi$  is contained in  $\text{Ker}(\Omega_M)$ . It therefore follows from Proposition 2.12 that  $\text{Ker}(\Omega_M)$  is isomorphic to  $M$ .  $\square$

## 5. UNIVERSAL STATES

Throughout this section we still use the same notation and terminology as in Section 2. Namely we assume that  $\mathcal{A}$  and  $\mathcal{I}$  are classes of  $\mathcal{L}$ -structures satisfying the assumptions of Theorem 2.8 such that  $\mathcal{A}$  has enough injectives from  $\mathcal{I}$  with modulus  $\varpi$ .

**5.1. Kubiś universal projections.** In [64, §4.1] Kubiś constructs, for any separable Lindenstrauss space  $Y$ , a projection  $\Omega_{\mathbb{G}}^Y$  of norm 1 on the Gurarij space  $\mathbb{G}$  with the following properties:

- the range of  $\Omega_{\mathbb{G}}^Y$  is isometrically isomorphic to  $Y$ , and the kernel of  $\Omega_{\mathbb{G}}^Y$  is isometric to  $\mathbb{G}$ ;
- for any separable Banach space  $X$ , and contractive linear mapping  $\phi : X \rightarrow \mathbb{G}$  whose range is contained in the range of  $\Omega_{\mathbb{G}}^Y$ , there exists an embedding  $\eta : X \rightarrow \mathbb{G}$  such that  $\phi = \Omega_{\mathbb{G}}^Y \circ \eta$ .

The existence of such a projection implies that  $\mathbb{G}$  is topologically isomorphic to  $\mathbb{G} \oplus X$  for any separable Lindenstrauss space  $X$ . (Recall that two Banach spaces  $X, Y$  are topologically isomorphic if there exists a bounded linear isomorphism from  $X$  to  $Y$ .) It follows that if  $Z$  is a separable Lindenstrauss space that contains a complemented subspace isomorphic to  $\mathbb{G}$ , then  $Z$  is topologically isomorphic to  $\mathbb{G}$ . Hence  $\mathbb{G}$  is also topologically isomorphic to  $\mathbb{G} \otimes X$  for any separable Lindenstrauss space  $X$ . A similar result is obtained in [16, Section 6] for  $p \in (0, 1]$  for the  $p$ -Gurarij space  $\mathbb{G}_p$ , which is the Fraïssé limit of the class of finite-dimensional  $p$ -Banach spaces.

In this section we will prove general results that imply the following characterization of Kubiś' universal projection universal projection  $\Omega_{\mathbb{G}}^Y$ ; see Subsection 6.1.

**Theorem 5.1.** *Fix a separable Lindenstrauss space. Suppose that  $T : \mathbb{G} \rightarrow Y$  is a linear map of norm at most 1,  $N$  is the kernel of  $T$ , and  $H = N^\perp \cap \mathbb{L}$ . The following assertions are equivalent:*

- (1)  *$T$  is a quotient mapping and  $N$  is a nonzero  $M$ -ideal of  $\mathbb{G}$ ;*
- (2)  *$T$  is a quotient mapping and  $H$  is a closed proper biface of  $\mathbb{L}$  symmetrically affinely homeomorphic to  $\text{Ball}(Y^*)$ ;*
- (3) *whenever  $E \subset F$  are finite-dimensional Banach spaces,  $f : E \rightarrow Y$  is a linear isometry,  $s : F \rightarrow Y$  is a linear map of norm at most 1 such that  $T \circ f = s$ , and  $\varepsilon > 0$ , there exists a linear isometry  $\hat{f} : F \rightarrow Y$  such that  $\|T \circ \hat{f} - s\| < \varepsilon$ .*

*The set of operators satisfying the equivalent conditions above is a dense  $G_\delta$  subset of the space  $\text{Ball}(B(\mathbb{G}))$  of linear maps from  $\mathbb{G}$  to  $Y$  of norm at most 1, and forms a single orbit under the action  $\text{Aut}(\mathbb{G}) \curvearrowright \text{Ball}(B(\mathbb{G}))$ ,  $(\alpha, S) \mapsto S \circ \alpha^{-1}$ . If  $\Omega_{\mathbb{G}}^Y : \mathbb{G} \rightarrow Y$  is such an operator, then the kernel of  $\Omega_{\mathbb{G}}^Y$  is isometrically isomorphic to  $\mathbb{G}$ . In other words the sequence*

$$0 \longrightarrow \mathbb{G} \longrightarrow \mathbb{G} \xrightarrow{\Omega_{\mathbb{G}}^Y} Y \longrightarrow 0$$

*where the first arrow is a linear isometry, is exact. Furthermore  $\Omega_{\mathbb{G}}^Y$  is a universal linear map of norm at most 1 from a separable Banach space to  $Y$ , in the sense that any if  $E$  is a separable Banach space, and  $L : E \rightarrow Y$  is a linear map of norm at most 1, then there exists a linear isometry  $\eta : E \rightarrow \mathbb{G}$  such that  $\Omega_{\mathbb{G}}^Y \circ \eta = L$ . In particular  $\Omega_{\mathbb{G}}^Y$  can be regarded as a projection of norm 1 onto an isometric copy of  $Y$  inside  $\mathbb{G}$ .*

The analog of Theorem 5.1 in the case of the Poulsen system holds as well.

**Theorem 5.2.** *Fix  $K$  is a metrizable Choquet simplex. Suppose that  $T : A(\mathbb{P}) \rightarrow A(K)$  is a unital positive linear map,  $N$  is the kernel of  $T$ , and  $H = N^\perp \cap \mathbb{P}$ . The following assertions are equivalent:*

- (1)  *$T$  is a quotient mapping and  $N$  is a nonzero  $M$ -ideal of  $A(\mathbb{P})$ ;*
- (2)  *$T$  is a quotient mapping and  $H$  is a closed proper face of  $\mathbb{P}$  affinely homeomorphic to  $K$ ;*
- (3) *whenever  $E \subset F$  are finite-dimensional function systems,  $f : E \rightarrow A(K)$  is a linear isometry,  $s : F \rightarrow A(K)$  is a unital linear function system such that  $T \circ f = s$ , and  $\varepsilon > 0$ , there exists a unital linear isometry  $\hat{f} : F \rightarrow A(K)$  such that  $\|T \circ \hat{f} - s\| < \varepsilon$ .*

The set of operators satisfying the equivalent conditions above is a dense  $G_\delta$  subset of the space  $\text{UP}(A(\mathbb{P}), A(K))$  of unital positive linear maps from  $A(\mathbb{P})$  to  $A(K)$ , and forms a single orbit under the action  $\text{Aut}(A(\mathbb{P})) \curvearrowright \text{UP}(A(\mathbb{P}), A(K))$ ,  $(\alpha, S) \mapsto S \circ \alpha^{-1}$ . If  $\Omega_{A(\mathbb{P})}^{A(K)} : A(\mathbb{P}) \rightarrow A(K)$  is such an operator, then

$$\left\{x \in A(\mathbb{P}) : \Omega_{A(\mathbb{P})}^{A(K)}(x) \text{ is a scalar multiple of the identity}\right\}$$

is unittally isometrically isomorphic to  $A(\mathbb{P})$ . Furthermore  $\Omega_{A(\mathbb{P})}^{A(K)}$  is a universal unital positive linear map from a separable function system to  $A(K)$ , in the sense that any if  $A(T)$  is a separable function system, and  $L : A(T) \rightarrow A(K)$  is a unital positive linear map, then there exists a unital linear isometry  $\eta : A(T) \rightarrow A(\mathbb{P})$  such that  $\Omega_{A(\mathbb{P})}^{A(K)} \circ \eta = L$ . In particular  $\Omega_{A(\mathbb{P})}^{A(K)}$  can be regarded as a projection onto a unital isometric copy of  $A(K)$  inside  $A(\mathbb{P})$ .

The noncommutative analogs of Theorem 5.1 and Theorem 5.2 hold as well; see Subsection 8.1 and Subsection 8.2.

**Theorem 5.3.** Fix a separable nuclear operator space  $Y$  and let  $\text{NG}$  be the noncommutative Gurarij space. There exists a linear complete contraction  $\Omega_{\text{NG}}^Y : \text{NG} \rightarrow Y$  such that if  $E$  is a separable Banach space, and  $L : E \rightarrow Y$  is a completely contractive linear map, then there exists a linear complete isometry  $\eta : E \rightarrow \text{NG}$  such that  $\Omega_{\text{NG}}^Y \circ \eta = L$ . Furthermore  $\Omega_{\text{NG}}^Y$  is generic, in the sense that the orbit of  $\Omega_{\text{NG}}^Y$  inside the space  $\text{Ball}(CB(\text{NG}))$  of completely contractive linear maps from  $\text{NG}$  to  $Y$  under the action  $\text{Aut}(\text{NG}) \curvearrowright \text{Ball}(CB(\text{NG}))$ ,  $(\alpha, S) \mapsto S \circ \alpha^{-1}$  is a dense  $G_\delta$  set. The kernel of  $\Omega_{\text{NG}}^Y$  is isometrically isomorphic to  $\text{NG}$ . In other words the sequence

$$0 \longrightarrow \text{NG} \longrightarrow \text{NG} \xrightarrow{\Omega_{\text{NG}}^Y} Y \longrightarrow 0$$

where the first arrow is a linear isometry, is exact. A completely contractive linear map  $T : \text{NG} \rightarrow Y$  belongs to the  $\text{Aut}(\text{NG})$ -orbit of  $\Omega_{\text{NG}}^Y$  if and only if it satisfies the following property: whenever  $E \subset F$  are finite-dimensional operator spaces,  $f : E \rightarrow Y$  is a linear complete isometry,  $s : F \rightarrow Y$  is a completely contractive linear map such that  $T \circ f = s$ , and  $\varepsilon > 0$ , there exists a linear complete isometry  $\hat{f} : F \rightarrow Y$  such that  $\|T \circ \hat{f} - s\| < \varepsilon$ .

Two operator spaces  $X, Y$  are completely isomorphic if there exists a completely bounded linear isomorphism from  $X$  to  $Y$ . A subspace of an operator space is completely complemented if it is the range of a completely bounded projection.

**Corollary 5.4.** The noncommutative Gurarij space  $\text{NG}$  is completely isomorphic to  $\text{NG} \oplus Y$  for any separable nuclear operator space  $Y$ . If a nuclear operator space contains a completely complemented subspace isomorphic to  $\text{NG}$ , then it is isomorphic to  $\text{NG}$ . In particular  $\text{NG}$  is isomorphic to  $\text{NG} \otimes Y$  for any separable nuclear operator space  $X$ .

*Proof.* By Theorem 5.3 one has an exact sequence of completely contractive maps

$$0 \longrightarrow \text{NG} \longrightarrow \text{NG} \longrightarrow Y \longrightarrow 0$$

where the second map is a complete isometry. It follows that  $\text{NG}$  is completely isomorphic to  $\text{NG} \oplus Y$ . The other assertions follow as in the proof of [16, Corollary 6.6].  $\square$

**Theorem 5.5.** Fix a separable nuclear operator system  $Y$  and let  $\text{NP}$  be the noncommutative Poulsen simplex, with associated operator system  $A(\text{NP})$ . There exists a unital completely positive map  $\Omega_{A(\text{NP})}^Y : A(\text{NP}) \rightarrow Y$  such that if  $X$  is a separable operator system, and  $L : X \rightarrow Y$  is a unital completely positive linear map, then there exists a unital linear complete isometry  $\eta : X \rightarrow A(\text{NP})$  such that  $\Omega_{A(\text{NP})}^Y \circ \eta = L$ . Furthermore  $\Omega_{A(\text{NP})}^Y$  is generic, in the sense that the  $\text{Aut}(A(\text{NP}))$ -orbit of  $\Omega_{A(\text{NP})}^Y$  inside the space  $\text{UCP}(A(\text{NP}), Y)$  of unital completely positive linear maps from  $A(\text{NP})$  to  $Y$  under the action  $\text{Aut}(A(\text{NP})) \curvearrowright \text{UCP}(A(\text{NP}), Y)$ ,  $(\alpha, S) \mapsto S \circ \alpha^{-1}$  is a dense  $G_\delta$  set. The set

$$\{x \in A(\text{NP}) : \Omega_{A(\text{NP})}^Y(x) \text{ is a scalar multiple of the identity}\}$$

is unittally completely isometrically isomorphic to  $A(\text{NP})$ . A unital completely positive map  $T : A(\text{NP}) \rightarrow Y$  belongs to the  $\text{Aut}(A(\text{NP}))$ -orbit of  $\Omega_{A(\text{NP})}^Y$  if and only if it satisfies the following property: whenever  $E \subset F$  are finite-dimensional operator systems,  $f : E \rightarrow Y$  is a unital linear complete isometry,  $s : F \rightarrow Y$  is a unital completely positive such that  $T \circ f = s$ , and  $\varepsilon > 0$ , there exists a unital linear complete isometry  $\hat{f} : F \rightarrow Y$  such that  $\|T \circ \hat{f} - s\| < \varepsilon$ .

The universal operators  $\Omega_{\mathbb{G}}$ ,  $\Omega_{\mathbb{P}}$ ,  $\Omega_{\mathbb{NG}}$ , and  $\Omega_{\mathbb{NP}}$  from Theorem 4.1, Theorem 4.2, Theorem 4.3, and Theorem 4.4 can be obtained from Theorem 5.1, Theorem 5.2, Theorem 5.3, and Theorem 5.5 in the particular case when  $Y = \mathbb{G}$ ,  $Y = A(\mathbb{P})$ ,  $Y = \mathbb{NG}$ , and  $Y = A(\mathbb{NP})$ , respectively.

**5.2. States as structures.** Fix an approximately injective separable structure  $R$  in  $\mathcal{A}$ ; see Definition 3.2. An  $R$ -state is a morphism  $s : X_s \rightarrow R$  from a structure  $X_s$  in  $\mathcal{A}$  to  $R$ . The terminology comes from the case of function systems, for which a state is a unital positive linear functional; see §6.3. We regard  $R$ -states as structures in a category  $\mathcal{A}_R$ . A morphism from  $s$  to  $t$  is a morphism  $f : X_s \rightarrow X_t$  in  $\mathcal{A}$ . We do not require that  $t \circ f = s$ . We consider  $R$ -states as structures in a language  $\mathcal{L}_R$  containing two sorts  $D_X$  and  $D_R$  and a function symbol  $D_X \rightarrow D_R$ . Furthermore for any  $n$ -ary function symbol  $f$  in  $\mathcal{L}$  one has an  $n$ -ary function symbols  $f_X : D_X^n \rightarrow D_X$  and an  $n$ -ary function symbol  $f_R : D_R^n \rightarrow D_R$ . Similarly for any  $n$ -ary relation symbol  $B$  in  $\mathcal{L}$  one has an  $n$ -ary relation symbol  $B_X : D_X^n \rightarrow \mathbb{R}$  and an  $n$ -ary relation symbol  $B_R : D_R^n \rightarrow \mathbb{R}$ . If  $X$  is a separable structure in  $\mathcal{A}$ , then the space  $S(X, R)$  of  $R$ -states on  $X$  endowed with the topology of pointwise convergence is a Polish space. The Polish group  $\text{Aut}(X)$  acts continuously on  $S(X, R)$  by  $(\alpha, s) \mapsto s \circ \alpha^{-1}$ .

**5.3. The generic state.** Suppose that  $X, \hat{X}, Y$  are structures in  $\mathcal{A}$ ,  $\hat{s}$  and  $t$  are  $R$ -states on  $\hat{X}$  and  $Y$  respectively, and  $f : X \rightarrow Y$  and  $\phi : X \rightarrow \hat{X}$  are morphisms such that  $I(\phi) < \delta$  and  $d(\hat{s} \circ \phi, t \circ f) \leq \varpi(\delta)$ . Let  $\hat{Y}$  be the approximate pushout of  $f$  and  $\phi$  defined as in Lemma 2.10, with canonical morphism  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  and embedding  $j : Y \rightarrow \hat{Y}$ . It follows from the universal property of the approximate pushout that there exists a (unique)  $R$ -state  $\hat{t}$  on  $\hat{Y}$  such that  $\hat{t} \circ \hat{f} = \hat{s}$  and  $\hat{t} \circ j = t$ . Again a similar argument applies the approximate pushouts over a tuple as in Lemma 3.1.

Using this observation one can show that the states  $s$  in  $\mathcal{A}_R$  such that  $X_s$  is a finitely generated structure form a Fraïssé class. The corresponding limit  $\Omega_M^R$  is an  $R$ -state on the Fraïssé limit  $M$  of the class of finitely generated structures in  $\mathcal{A}$ , as it can be verified using uniqueness of the limit and approximate injectivity of  $R$ . Furthermore if  $s$  is an  $R$ -state on  $M$ , then the following assertions are equivalent:

- (1) There exists an automorphism  $\alpha$  of  $M$  such that  $s \circ \alpha = \Omega_M^R$ ;
- (2) Whenever  $\phi : E \rightarrow F$  is a morphism between finitely generated structures in  $\mathcal{A}$  such that  $I(\phi) < \delta$ ,  $t$  is an  $R$ -state on  $F$ , and  $f : E \rightarrow M$  is a morphism such that  $d(t \circ \phi, s \circ f) < \varpi(\delta)$ , there exists an embedding  $g : F \rightarrow M$  such that  $s \circ g = t$  and  $d(g \circ \phi, f) < \varpi(\delta)$ ;
- (3) Whenever  $E, F$  are finitely generated structures in  $\mathcal{A}$  such that  $F \in \mathcal{I}$ ,  $t$  is an  $R$ -state of  $F$ ,  $\phi : E \rightarrow F$  and  $f : E \rightarrow M$  are embeddings such that  $t \circ \phi = s \circ f$ , and  $\varepsilon > 0$ , there exists an embedding  $g : F \rightarrow M$  such that  $s \circ g = t$  and  $d(g \circ \phi, f) < \varepsilon$ ;
- (4) For any finitely generated structure  $E$  in  $\mathcal{A}$ ,  $R$ -state  $t$  on  $E$ , and morphisms  $f : E \rightarrow M$  and  $\phi : E \rightarrow M$  such that  $I(\phi) < \delta$ ,  $I(f) < \delta$ , and  $d(s \circ \phi, s \circ f) < \varpi(\delta)$ , there exists an automorphism  $\beta$  of  $M$  such that  $d(\beta \circ \phi, f) < \varpi(\delta)$  and  $s \circ \beta = s$ ;
- (5) For any finite tuple  $\bar{b}$  in  $M$ , morphisms  $f : \langle \bar{b} \rangle \rightarrow M$  and  $\phi : \langle \bar{b} \rangle \rightarrow M$  such that  $I(f) < \delta$ ,  $I(\phi) < \delta$ ,  $s \circ \phi \approx_{\bar{a}, \delta} f$ , there exists an automorphism  $\beta$  of  $M$  such that  $\beta \circ \phi \approx_{\bar{a}, \varpi(\delta)} f$  and  $s \circ \beta = s$ ;
- (6) the same as (5) where moreover  $\bar{b} \in M_0$  and  $f(\bar{b}), \phi(\bar{b}) \in B_0$  for some fixed countable fundamental subsets  $M_0$  of  $M$  and  $B_0$  of  $B$  as in Definition 2.5.

Such a characterization in particular shows that the set  $\{\Omega_M^R \circ \alpha : \alpha \in \text{Aut}(M)\}$  is a dense  $G_\delta$  subset of the space of  $R$ -states of  $M$ . It is not difficult to verify using the universal property characterizing the universal state  $\Omega_M^R$  and the universal operator  $\Omega_M$  as in Subsection 4.3 that  $\Omega_M$  and  $\Omega_M^R$  for  $R = M$  have the same  $\text{Aut}(M)$ -orbit. In the case of rigid  $\mathcal{I}$ -structures as in Definition 3.6,  $\Omega_M^R$  admits the following further characterization, which can be proved similarly as Proposition 3.7.

**Proposition 5.6.** *Let  $X$  be a rigid  $\mathcal{I}$ -structure, and  $s$  be an  $R$ -state on  $X$ . If for any  $\varepsilon > 0$ , structures  $A, \hat{A} \in \mathcal{I}$ ,  $R$ -state  $t$  on  $\hat{A}$ , and embeddings  $\phi : A \rightarrow \hat{A}$  and  $f : A \rightarrow X$  such that  $t \circ \phi = s \circ f$ , there exists an embedding  $\hat{f} : \hat{A} \rightarrow X$  such that  $d(s \circ \hat{f}, t) < \varepsilon$  and  $d(\hat{f} \circ \phi, f) < \varepsilon$ , then there exists an isomorphism  $\alpha : X \rightarrow M$  such that  $\Omega_M^R \circ \alpha = s$ .*

One can prove similarly as in Proposition 4.7 that under the same assumptions of Proposition 4.7 the morphism  $\Omega_R$  is surjective, and  $\text{Ker}(\Omega_R)$  is isomorphic to  $M$ . Universality of the Fraïssé limit implies that if  $X$  is a separable structure in  $\mathcal{A}$  and  $s$  is an  $R$ -state on  $X$ , then there exists an embedding  $\phi : X \rightarrow M$  such that  $\Omega_M^R \circ \phi = s$ . In particular letting  $X = R$  and  $s$  be the identity map of  $R$  one can conclude that there exists an embedding  $\eta_R : R \rightarrow X$  such that  $\Omega_M^R \circ \eta_R$  is the identity map of  $R$ . This also implies that  $\Omega_M^R$  is surjective. Defining  $\rho_R$  to be  $\eta_R \circ \Omega_M^R$  gives a retraction of  $M$  onto a substructure of  $M$  isomorphic to  $R$ . This shows that  $R$  is isomorphic to a retract of  $M$ , which is the content of Theorem 3.3. Furthermore  $\rho_R$  is a *universal retraction*

in the following sense. If  $X$  is a separable structure in  $\mathcal{A}$  and  $s$  is a state on  $X$  whose range is contained in the range of  $\rho_R$ , then there exists an embedding  $\psi : X \rightarrow M$  such that  $\rho_R \circ \psi = s$ .

**Remark 5.7.** A further back-and-forth argument together with Condition (4) in the characterization of the universal state  $\Omega_M^R$  shows that any automorphism of  $R$  “lifts” to an automorphism of  $M$ . This means that if  $\sigma$  is an automorphism of  $R$ , then there exists an automorphism  $\hat{\sigma}$  of  $M$  such that  $\sigma \circ \Omega_M^R = \Omega_M^R \circ \hat{\sigma}$ .

A similar construction to the one above is performed in [64, §4.1] in the case of Banach spaces and, more generally, in [16, Section 6] in the case of  $p$ -Banach spaces for every  $p \in (0, 1]$ . The case of Banach spaces is subsumed by the above general results; see §6.1. The case of  $p$ -Banach spaces for  $p \in (0, 1)$  does not fit in the framework of this paper, since no nontrivial  $p$ -Banach space for  $p \in (0, 1)$  is injective [16, Proposition 5.2]. However, one can consider a generalization of the assumptions considered in this paper, where the structures in the class  $\mathcal{I}$  are not assumed to be injective, but only approximately injective as in Definition 3.2. In this more general framework one can recover the main results of [16] concerning  $p$ -Banach spaces for arbitrary  $p \in (0, 1]$ .

**5.4. The  $\text{Aut}(M)$ -space  $S(M, R)$ .** The automorphism group  $\text{Aut}(M)$  of  $M$  is a Polish group when endowed with the topology of pointwise convergence. Also  $S(M, R)$  is a Polish space endowed with the topology of pointwise convergence.

We regard  $S(M, R)$  as a uniform  $\text{Aut}(M)$ -space endowed with the uniformity generated by the sets of the form

$$\{(s_0, s_1) \in S(M, R) \times S(M, R) : d(s_0(x), s_1(x)) < \varepsilon\}$$

for  $x \in M$  and  $\varepsilon > 0$ . The action of  $\text{Aut}(M)$  on  $S(M, R)$  is defined by  $(\alpha, s) \mapsto s \circ \alpha^{-1}$ . By completeness of  $R$ , the uniform space  $S(M, R)$  is complete as well. Furthermore  $S(M, R)$  is compact whenever  $R$  is compact. Let  $\text{Aut}(M, \Omega_M^R)$  be the stabilizer of  $\alpha$ , i.e. the group of automorphisms  $\alpha$  of  $M$  such that  $\Omega_M^R \circ \alpha = \Omega_M^R$ . We also regard  $\text{Aut}(M)/\text{Aut}(M, \Omega_M^R)$  as a uniform  $\text{Aut}(M)$ -space endowed with the quotient of the right uniformity on  $\text{Aut}(M)$  and the canonical action by translation. The sets of the form

$$\{(\alpha_0, \alpha_1) \in \text{Aut}(M) \times \text{Aut}(M) : d(\alpha_0^{-1}(x), \alpha_1^{-1}(x)) < \varepsilon\}$$

form a basis of entourages for the right uniformity on  $\text{Aut}(M)$ .

One can define the map  $\pi : \text{Aut}(M)/\text{Aut}(M, \Omega_M^R) \rightarrow S(M, R)$  mapping the left coset of  $\text{Aut}(M, \Omega_M^R)$  with respect to  $\alpha$  to  $s \circ \alpha$ . Clearly  $\pi$  is an injective  $\text{Aut}(M)$ -equivariant uniformly continuous map. Furthermore by genericity of  $\Omega_M^R$ ,  $\pi$  has dense image. We claim that  $\pi^{-1}$  is uniformly continuous as well. Indeed suppose that  $\bar{a}$  is a finite tuple in  $M$  and  $\varepsilon > 0$ . If  $\alpha, \beta$  are automorphisms such that  $\Omega_M^R \circ \alpha^{-1} \approx_{\bar{a}, \varepsilon} \Omega_M^R \circ \beta^{-1}$ , then by Condition (5) in the characterization of  $\Omega_M^R$  there exists  $\gamma \in \text{Aut}(M, \Omega_M^R)$  such that  $(\alpha \circ \gamma)^{-1} \approx_{\bar{a}, \varpi(\delta)} \beta^{-1}$ . This concludes the proof that  $\pi^{-1}$  is uniformly continuous. In particular this shows that the completion of  $\text{Aut}(M)/\text{Aut}(M, \Omega_M^R)$  is  $\text{Aut}(M)$ -equivariantly uniformly isomorphic to  $S(M, R)$ .

Recall that, if  $G$  is a topological group, then a uniform  $G$ -space is *minimal* if every orbit of  $G$  is dense. We can provide a reformulation of the assertion that  $S(M, R)$  is a minimal  $\text{Aut}(M)$ -space in terms of the Fraïssé class  $\mathcal{C}$ .

**Proposition 5.8.** *Consider the following assertions:*

- (1) *For every tuple  $\bar{a}$  in  $M$ ,  $s \in S(M, R)$ , and  $\varepsilon > 0$ , there exists  $B \in \mathcal{I}$  such that for any  $t \in S(B, R)$  there exists a morphism  $\phi : \langle \bar{a} \rangle \rightarrow B$  such that  $I(\phi) < \varepsilon$  and  $d((t \circ \phi)(\bar{a}), s(\bar{a})) < \varepsilon$ ;*
- (2) *for every tuple  $\bar{a}$  in  $M$  such that  $\langle \bar{a} \rangle \in \mathcal{I}$ ,  $s \in S(M, R)$ , and  $\varepsilon > 0$ , there exists a finitely generated substructure  $B$  of  $M$  such that for any  $t \in S(M, R)$  there exists a morphism  $\phi : \langle \bar{a} \rangle \rightarrow B$  such that  $I(\phi) < \varepsilon$  and  $d((t \circ \phi)(\bar{a}), s(\bar{a})) < \varepsilon$ ;*
- (3) *the action  $\text{Aut}(M) \curvearrowright S(M, R)$  is minimal.*

*Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). If furthermore  $R$  is compact, then (3) $\Rightarrow$ (1).*

*Proof.* The implication (1) $\Rightarrow$ (2) is obvious.

For (2) $\Rightarrow$ (3), suppose that  $s, t \in S(M, R)$ ,  $\bar{a}$  is a tuple in  $M$ , and  $\varepsilon > 0$ . We want to find  $\alpha \in \text{Aut}(M)$  such that  $d((s \circ \alpha)(\bar{a}), t(\bar{a})) < \varepsilon$ . Without loss of generality we can assume that  $\langle \bar{a} \rangle \in \mathcal{I}$ . The automorphism  $\alpha$  can then be obtained from the hypothesis using the stable homogeneity property of  $M$ .

We now assume that  $R$  is compact, and prove (3) $\Rightarrow$ (1). Suppose that  $\text{Aut}(M) \curvearrowright S(M, R)$  is minimal, but (1) does not hold. Thus for some tuple  $\bar{a}$  in  $M$ ,  $\varepsilon_0 > 0$ , and  $s_0 \in S(M, R)$ , for every  $B \in \mathcal{I}$  there exists  $t_B \in S(M, R)$  such that for every morphism  $\phi : \langle \bar{a} \rangle \rightarrow B$  such that  $I(\phi) < \varepsilon_0$  one has that  $d((t_B \circ \phi)(\bar{a}), s_0(\bar{a})) \geq \varepsilon_0$ . Without loss of generality we can assume that  $\bar{a}$  is a basic tuple. Let  $\mathcal{B}$  be the set of pairs  $(B, \delta)$  such that  $B \in \mathcal{I}$  and  $\delta > 0$ . For every  $\eta > 0$  and tuple  $\bar{b}$  in  $M$ , let  $\mathcal{B}_{\bar{b}, \eta}$  be the set of  $(B, \delta) \in \mathcal{B}$  such that  $\bar{b} \subset_{\eta} B$  and  $\delta < \eta$ .

Observe that the collection of subsets  $\mathcal{B}_{\bar{b},\eta}$  of  $\mathcal{B}$  where  $\bar{b}$  varies among the finite tuples in  $M$  and  $\eta > 0$  has the finite intersection property. Therefore there exists an ultrafilter  $\mathcal{U}$  on  $\mathcal{B}$  that contains the set  $\mathcal{B}_{\bar{b},\eta}$  for every tuple  $\bar{b}$  in  $M$  and  $\eta > 0$ . Fix  $x \in M$  and  $(B, \eta) \in \mathcal{B}$ . Define  $t_{B,\eta}(x) = t_B(x)$  for any  $M \supset B \in \mathcal{I}$  such that  $x \in_\eta B$ . Finally let  $t(x)$  be the limit according to  $\mathcal{U}$  of the function  $(B, \eta) \mapsto t_{B,\eta}(x)$ . This defines an element  $t$  of  $S(M, R)$ . By minimality of the action  $\text{Aut}(M) \curvearrowright S(M, R)$ , for every  $\delta > 0$  there exists  $\alpha \in \text{Aut}(M)$  such that  $d((t \circ \alpha)(\bar{a}), \bar{a}) < \delta$ . Using the hypotheses on basic sequences from Subsection 2.2, this easily leads to a contradiction with our assumption.  $\square$

**Corollary 5.9.** *If for every  $A \in \mathcal{I}$ ,  $s \in S(A, R)$ , and  $\varepsilon > 0$ , there exists  $B \in \mathcal{I}$  such that for any  $t \in S(B, R)$  there exists a morphism  $\phi : A \rightarrow B$  such that  $I(\phi) < \varepsilon$  and  $d(t \circ \phi, s) < \varepsilon$ , then  $\text{Aut}(M) \curvearrowright S(M, R)$  is minimal.*

Suppose that  $G$  is a topological group, and  $X$  is a compact space. A continuous action of  $G \curvearrowright X$  is called *proximal* if for every entourage  $U$  of the unique compatible uniformity of  $X$  and  $x, y \in X$  there exists  $g \in G$  such that  $(gx, gy) \in U$  [46, §I.1]. More generally we call a uniform  $G$ -space  $X$  proximal if it satisfies the same property where  $U$  is an entourage of the given uniformity of  $X$ . The following characterization of classes for which the action  $\text{Aut}(M) \curvearrowright S(M, R)$  is proximal is an immediate consequence of stable homogeneity of the limit  $M$  and our assumptions on basic sequences.

**Proposition 5.10.** *The following assertions are equivalent:*

- (1) *For every tuple  $\bar{a}$  in  $M$ ,  $s, t \in S(M, R)$ , and  $\varepsilon > 0$ , there exists  $B \in \mathcal{I}$  and a morphism  $\phi : \langle \bar{a} \rangle \rightarrow B$  such that  $I(\phi) < \varepsilon$  and  $d((t \circ \phi)(\bar{a}), (s \circ \phi)(\bar{a})) < \varepsilon$ ;*
- (2) *for every tuple  $\bar{a}$  in  $M$  such that  $\langle \bar{a} \rangle \in \mathcal{I}$ ,  $s, t \in S(M, R)$ , and  $\varepsilon > 0$ , there exists a finitely generated structure  $B$  in  $\mathcal{A}$  and a morphism  $\phi : \langle \bar{a} \rangle \rightarrow B$  such that  $I(\phi) < \varepsilon$  and  $d((t \circ \phi)(\bar{a}), s(\bar{a})) < \varepsilon$ ;*
- (3) *the action  $\text{Aut}(M) \curvearrowright S(M, R)$  is proximal.*

## 6. EXAMPLES

In this section we explain how many classes of structures fit into the framework of Sections 2, 3, 4, 5.

**6.1. Real Banach spaces.** In this subsection we assume all the Banach spaces to be over the real numbers. Suppose that  $\mathcal{L}$  is the language containing binary function symbols  $f_{\lambda,\mu}$  for  $\lambda, \mu \in \mathbb{Q}$  such that  $|\lambda| + |\mu| \leq 1$ . We can identify a Banach space  $X$  with its unit ball  $\text{Ball}(X)$ , which is naturally an  $\mathcal{L}$ -structure where the interpretation of  $f_{\lambda,\mu}$  is the function  $(x, y) \mapsto \lambda x + \mu y$ . Under this identification, the morphisms according to Definition 2.1 are precisely the restriction to the unit ball of bounded linear maps of norm at most 1. Indeed suppose that  $T : \text{Ball}(X) \rightarrow \text{Ball}(Y)$  is a morphism. One can extend  $T$  to a linear map from  $X$  to  $Y$  of norm at most 1 by setting  $T(x) = \|x\| T(x/\|x\|)$  for any nonzero  $x \in X$ . Conversely it is clear that if  $T : X \rightarrow Y$  is a bounded linear map with  $\|T\| \leq 1$  then the restriction of  $T$  to  $\text{Ball}(X)$  is a morphism. We can therefore identify morphisms with bounded linear maps with norm at most one. If  $T : X \rightarrow Y$  is a bounded linear map of norm at most 1 and  $0 \leq \delta \leq 1$ , then  $I(T) \leq \delta$  as in Definition 2.3 if and only if  $\|Tx\| \geq \|x\| - \delta$  whenever  $\|x\| \leq 2$ , which in turn happens if and only if  $T$  is injective and  $\|T^{-1}\| \leq 1 + \delta$ .

It follows from the geometric version of the Hahn-Banach theorem that if  $\bar{a}$  is a tuple in  $\text{Ball}(X)$  then the substructure generated by  $\bar{a}$  according to Definition 2.2 is the unit ball of the linear span of  $\bar{a}$  inside  $X$ . We declare a tuple  $\bar{a}$  to be a basic tuple if and only if it is linearly independent. A simple calculation shows that such a notion of basic tuple satisfies the requirements of Subsection 2.2.

Let  $\mathcal{I}$  be the collection of Banach spaces  $\ell_n^\infty$  for  $n \in \mathbb{N}$ , which are precisely the injective finite-dimensional Banach spaces. It is easy to verify that Conditions (1) and (2) of Subsection 2.3 hold in this context. This shows that the class of finite-dimensional Banach spaces is a Fraïssé class. The corresponding limit is the Gurarij space first constructed by Gurarij [50] and proved to be unique by Lusky [82]; see also [84].

The following well known fact is a consequence of classical results of Lindenstrauss [78], Lazar–Lindenstrauss [73, 74], and Michael–Pełczyński [88]:

**Fact 6.1.** *For a separable Banach space the following conditions are equivalent:*

- (1)  *$X$  is approximately injective according to Definition 3.2;*
- (2)  *$X$  is an  $\mathcal{I}$ -structure according to Definition 3.6;*
- (3)  *$X$  is a rigid  $\mathcal{I}$ -structure according to Definition 3.6*
- (4)  *$X$  is an isometric predual of an  $L^1$  space;*
- (5)  *$X$  is linearly isometric to the limit of an inductive sequence of finite-dimensional injective Banach spaces.*

When  $X$  satisfies the equivalent conditions of Fact 6.1, it is called a *Lindenstrauss space*. It follows from Theorem 3.3 that a separable Banach space is a Lindenstrauss space if and only if it is isometric to a 1-complemented subspace of  $\mathbb{G}$ . This recovers a classical result of Wojtaszczyk [109].

A Banach space  $X$  is existentially closed (resp. positively existentially close) if for any isometric inclusion  $X \subset Y$  and quantifier-free formula (resp. atomic formula)  $\varphi(x, b)$  for  $b \in \text{Ball}(X)$  one has that  $\inf_a \varphi(a, b)$  has the same value when  $a$  ranges in the unit ball of  $X$  or the unit ball of  $Y$ ; see [49, Subsection 4.4]. It is clear that Condition (6) of Proposition 2.12 can be expressed by a first order formula in the language of Banach spaces. Therefore Proposition 2.12 shows that the Gurarij Banach space is the unique separable model of its first order theory as well as the only separable existentially closed Banach space, a fact already proved in [9]. Applying stable homogeneity of  $\mathbb{G}$  and [8, Proposition 13.6] one can recover the following result from [9]: the theory of  $\mathbb{G}$  admits elimination of quantifiers, and it is the model completion of the theory of Banach spaces. Finally the characterization of Lindenstrauss spaces mentioned above shows that a separable Banach space  $X$  is Lindenstrauss if and only if it is positively existentially closed.

**Definition 6.2.** *A compact absolutely convex set is a compact subset  $K$  of a real locally convex topological vector space with the property that  $\lambda x + \mu y \in K$  whenever  $x, y \in K$  and  $\lambda, \mu \in \mathbb{R}$  are such that  $|\lambda| + |\mu| \leq 1$ . If  $K$  is a compact absolutely convex set and  $F \subset K$ , then the absolutely convex hull of  $F$  is the smallest absolutely convex subset of  $K$  containing  $F$ .*

Let  $\sigma : K \rightarrow K$  be the involution  $p \mapsto -p$ . A function  $f : K \rightarrow \mathbb{R}$  is *symmetric* if  $f \circ \sigma = -f$ . More generally a function between compact absolutely convex sets is symmetric if it commutes with the involution. Similarly, a signed Borel measure  $\mu$  on  $K$  is *symmetric* if the pushforward  $\sigma\mu$  of  $\mu$  under  $\sigma$  is equal to  $-\mu$ .

If  $X$  is a Banach space, then the unit ball  $\text{Ball}(X^*)$  of the dual space of  $X$  is a compact absolutely convex set. Suppose that  $K$  is a compact absolutely convex set. We denote by  $A_\sigma(K)$  the space of continuous symmetric affine functions from  $K$  to  $\mathbb{R}$ . The map from  $K$  to  $\text{Ball}(A_\sigma(K)^*)$  mapping  $p$  to the evaluation functional at  $p$  is an affine symmetric homeomorphism [72, Lemma 1]. Furthermore the assignment  $K \mapsto A_\sigma(K)$  is a contravariant equivalence of categories from the category of Banach spaces and linear contractive maps to the category of compact absolutely convex sets and continuous symmetric affine functions. In the following we will assume all the Banach spaces to be separable, and all the compact absolutely convex sets to be metrizable.

**Definition 6.3.** *A Lazar simplex is a compact absolutely convex set that is symmetrically affinely homeomorphic to  $\text{Ball}(X^*)$  for some Lindenstrauss space  $X$ .*

Lazar provided in [72]—see also [33, Theorem 3.2]—the following characterization of Lindenstrauss simplices in terms of representing measures, similar in spirit to the characterization of Choquet simplices in terms of representing probability measures: a compact absolutely convex set  $K$  is a Lazar simplex if and only if given any two boundary Borel probability measures  $\mu_1, \mu_2$  on  $K$  with the same barycenter one has that  $\mu_1 - \sigma\mu_1 = \mu_2 - \sigma\mu_2$  or, equivalently,  $\int f d\mu_1 = \int f d\mu_2$  for any  $f \in A_\sigma(K)$ . We call the Lazar simplex  $\text{Ball}(\mathbb{G})$  associated with the Gurarij space the *Lusky simplex*, and denote it by  $\mathbb{L}$ .

Suppose that  $X$  is a Banach space. Two elements  $p, q \in X^*$  are called *codirectional* [2] (or without cancellation [33]) if  $\|p + q\| = \|p\| + \|q\|$ . Several equivalent characterizations of codirectional functionals are provided in [2, Lemma 2.3] and [33, Lemma 4.1]. An *L-projection* is an idempotent map  $P : X^* \rightarrow X^*$  such that  $\|x\| = \|P(x)\| + \|x - P(x)\|$  for every  $x \in X^*$ . A subspace  $J$  of  $X^*$  is called an *L-ideal* if it is the range of an L-projection. When such an L-projection exists, it is necessarily unique [52, Proposition 2.1]. A subspace of a Banach space  $X$  is an *M-ideal* if its annihilator is an L-ideal of  $X^*$ . A complete survey on the theory of M-ideals and L-ideals can be found in [53].

**Definition 6.4.** *Suppose that  $K$  is a compact absolutely convex set. A subset  $H$  of  $K$  is a biface if it is convex and symmetric,  $\|p\|^{-1}p \in H$  whenever  $p \in H$  is nonzero, and if  $q_0, q_1 \in K$  are codirectional and  $q_0 + q_1 \in H$  one has that  $q_0, q_1 \in H$ .*

A biface of  $K$  is *trivial* if  $H = \{0\}$  and *proper* if  $H \neq K$ . When  $X$  is a Lindenstrauss space,  $K = \text{Ball}(X^*)$ , and  $H \subset K$  is a  $w^*$ -closed absolutely convex subset, then  $H$  is a biface if and only if it is the absolutely convex hull of a face of  $K$  [48], if and only if the linear span of  $H$  in  $X^*$  is an L-ideal [3, §6]. Furthermore in this case one has that  $J \cap K = H$  [74, Lemma 2.1].

Recall that a Banach space has the *metric approximation property* if its identity map is the pointwise limit of finite rank linear contractions. Clearly any Lindenstrauss space has the metric approximation property. The following proposition collects several equivalent characterizations of M-ideals and bifaces in Lindenstrauss spaces.

**Proposition 6.5.** *Assume that  $Z, X$  are separable Lindenstrauss spaces, and  $P : Z \rightarrow X$  is a quotient mapping. Let  $P^\dagger$  the corresponding dual map from  $X^*$  to  $Z^*$ . Let  $K$  be the Lazar simplex  $\text{Ball}(Z^*)$ , and  $H$  be the image of  $\text{Ball}(X^*)$  under  $P^\dagger$ . Let also  $N$  be the kernel of  $P$ , and  $N^\perp \subset Z^*$  be the annihilator of  $N$ . Observe that  $N^\perp$  coincides with the image of  $X^*$  under  $P^\dagger$ , as well as with the linear span of  $H$  inside  $Z^*$ . The following statements are equivalent:*

- (1)  $N$  is an  $M$ -ideal of  $X$ ;
- (2) whenever  $\varepsilon > 0$ ,  $E \subset F$  are finite-dimensional Banach spaces,  $g : F \rightarrow X$  is a linear contraction and  $f : E \rightarrow Z$  is a linear isometry such that  $P \circ f = g|_E$ , then there exists a linear contraction  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\| \leq \varepsilon$ ;
- (3) whenever  $\varepsilon > 0$ ,  $A$  is a separable Banach space with the metric approximation property,  $E \subset A$  is a finite-dimensional subspace, and  $f : E \rightarrow Z$  and  $g : A \rightarrow X$  are linear contractions such that  $\|P \circ f - g|_E\| < \varepsilon$ , then there exists a linear contraction  $\hat{g} : A \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\| < 6\varepsilon$ ;
- (4) for any subspace  $E$  of  $Z$ ,  $\varepsilon \geq 0$ , one has that  $\|P(x)\| \geq (1 - \varepsilon)\|x\|$  for any  $x \in E$  if and only if there exists a linear contraction  $\eta : X \rightarrow Z$  such that  $P \circ \eta$  is the identity map of  $X$  and  $\|\eta \circ P|_E - \text{id}_E\| \leq \varepsilon$ ;
- (5) for any  $\varepsilon > 0$ ,  $y \in Z$  and  $u \in N$  such that  $\|y\| = \|u\| = 1$ , there exists  $v \in Z$  such that  $\|P(v)\| \leq \varepsilon$  and  $\|v - y \pm u\| \leq 1 + \varepsilon$ ;
- (6)  $H$  is a biface of  $K$ .

*Proof.* In the proof we identify  $Z$  with  $A_\sigma(K)$  and  $X$  with  $A_\sigma(H)$ . Under these identifications  $P$  is just the restriction mapping  $A_\sigma(K) \rightarrow A_\sigma(H)$ ,  $f \mapsto f|_H$ . The equivalence of (6) and (1) is proved in [3, §6]. The equivalence of (6) and (5) is essentially [86, Proposition 3]. The implication (1) $\Rightarrow$ (3) can be proved similarly as [20, Theorem 2.6] using [20, Lemma 2.5]. We prove the other nontrivial implications below.

(2) $\Rightarrow$ (1) Suppose that  $y_1, y_2, y_3 \in \text{Ball}(N)$  and  $x \in \text{Ball}(Z)$  and  $\varepsilon > 0$ . In view of the equivalence (i) $\Leftrightarrow$ (iv) in [53, Theorem 2.2], it is enough to prove that there exists  $y \in \text{Ball}(N)$  such that  $\|x + y^{(\ell)} - y\| \leq 1 + \varepsilon$  for  $\ell \in \{1, 2, 3\}$ . Let  $E = \text{span}\{y_1, y_2, y_3, x\} \subset Z$ . Consider the Banach space  $F$  obtained from  $E \oplus \mathbb{R}$  and the collection of maps  $(z, \lambda) \mapsto \varphi(z) + \lambda s$  where  $\varphi : E \rightarrow \mathbb{R}$  is a linear contraction and  $s \in [-1, 1]$  is such that  $|\varphi(x + y^{(\ell)}) - s| \leq 1$  for  $\ell \in \{1, 2, 3\}$ . Define also the map  $g : F \rightarrow X$  by  $(z, \lambda) \mapsto P(z)$ . Observe that the canonical inclusion  $E \subset F$  is isometric and the map  $g$  is a contraction such that  $g|_E = P$ . Hence by hypothesis there exists a linear contraction  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - \iota_E\| \leq \varepsilon$ , where  $\iota_E : E \rightarrow Z$  is the inclusion map. The element  $y := \hat{g}(0, 1)$  is as desired.

(6) $\Rightarrow$ (4): Suppose that  $E \subset Z$  is a linear subspace such that  $\|P(x)\| \geq (1 - \varepsilon)\|x\|$  for every  $x \in E$ . Let  $k$  be an element of  $K$ . We observe that there exists  $h \in H$  such that  $\|(k - h)|_E\| \leq \varepsilon$ . The assumption implies that  $E \cap N = \{0\}$ . Define  $h \in (E + N)^*$  by setting  $h(e + n) = (1 - \varepsilon)k(e)$ . We have that  $|k(e)| \leq \|e\| \leq (1 - \varepsilon)^{-1}\|P(e)\| \leq (1 - \varepsilon)^{-1}\|e + n\|$ . Thus  $\|h\| \leq 1$  and hence it extends to a linear functional on  $X$  of norm at most 1 that belongs to  $H = \text{Ball}(X^*) \cap N^\perp$ . It is clear from the definition that  $\|(k - h)|_E\| \leq \varepsilon$ . Define the function defined by

$$\varphi : k \mapsto \{h \in H : \|(k - h)|_E\| \leq \varepsilon\}$$

for  $k \in K$ . Observe that  $\varphi$  satisfies the assumptions of [74, Theorem 2.2]. Hence there exists a continuous affine symmetric function  $Q : K \rightarrow H$  such that  $Q|_H$  is the identity map of  $H$  and  $\|(Q(k) - k)|_E\| \leq \varepsilon$  for every  $k \in K$ . One can thus define  $\eta : A_\sigma(H) \rightarrow A_\sigma(K)$  by  $\eta \mapsto \eta \circ Q$ .

(5) $\Rightarrow$ (6) Suppose that  $q_0, q_1 \in K$  and  $p \in H$  are such that  $\|q_0\| + \|q_1\| = \|p\|$ ,  $t \in (0, 1)$ , and  $tq_0 + (1 - t)q_1 = p$ . We want to prove that  $q_0 \in H$ . Fix  $u \in N$  of norm 1 and  $y \in Z$  of norm 1 such that  $p(y) = 1$ . Observe that  $q_0(y) = q_1(y) = p(y) = 1$ . It is enough to prove that  $q_0(u) \leq 3\varepsilon$ . By assumption there exists  $v \in Z$  such that  $\|P(v)\| \leq \varepsilon$  and  $\|v - y \pm u\| \leq 1 + \varepsilon$ . Then we have  $\|v - y\| \leq 1 + \varepsilon$ . Thus

$$q_0(v) \leq \varepsilon - q_1(v) \leq \varepsilon + (1 + \varepsilon) - q_0(y) \leq 2\varepsilon$$

and

$$q_0(u) = q_0(y + u - v) - q_0(y) - q_0(v) \leq 3\varepsilon.$$

This concludes the proof.  $\square$

**Remark 6.6.** *The equivalence of (1)–(3) in Proposition 6.5 holds even without the assumption that  $Z, X$  are Lindenstrauss spaces. Furthermore if  $H$  is a closed biface of a metrizable Lazar simplex  $K$  then the restriction mapping  $A_\sigma(K) \rightarrow A_\sigma(H)$ ,  $f \mapsto f|_H$  is automatically a complete quotient mapping by [74, Corollary 1].*

The equivalence of the conditions in Proposition 6.5 justifies the following definition.

**Definition 6.7.** *If  $X, Z$  are Lindenstrauss spaces, and  $P : Z \rightarrow X$  is a quotient linear mapping, then we say that  $P$  is a facial quotient if it satisfies any of the equivalent conditions of Proposition 6.5. A facial quotient is trivial if it is an isometric isomorphism.*

Let us now fix a separable Lindenstrauss space  $X$  and consider the generic operator  $\Omega_{\mathbb{G}}^X : \mathbb{G} \rightarrow X$  constructed as in Section 5. It follows from the characterization of the generic state from Subsection 5.3 together with Proposition 6.5 that  $\Omega_{\mathbb{G}}^X$  is a nontrivial facial quotient with kernel isometrically isomorphic to  $\mathbb{G}$ . Therefore any Lazar simplex is symmetrically affinely homeomorphic to a closed proper biface of  $\mathbb{L} = \text{Ball}(\mathbb{G})$  [86, Corollary 4]. In the rest of the section we will prove that, conversely, any nontrivial facial quotient  $P : \mathbb{G} \rightarrow X$  belongs to the  $\text{Aut}(\mathbb{G})$ -orbit of  $\Omega_{\mathbb{G}}^X$ .

Let us consider initially the case  $X = \mathbb{R}$ . In this case we have that  $\Omega_{\mathbb{G}}^{\mathbb{R}}$  is an extreme point of  $\mathbb{L} := \text{Ball}(\mathbb{G}^*)$ . Hence the extreme boundary of  $\mathbb{L}$  is dense in  $\mathbb{L}$ . We now want to observe that, conversely, any Lazar simplex with dense extreme boundary is symmetrically affinely homeomorphic to  $\mathbb{L}$ .

**Proposition 6.8** ([79, Theorem 6.1]). *Suppose that  $L$  is a nontrivial metrizable Lazar simplex with dense extreme boundary. Then  $L$  is symmetrically affinely homeomorphic to  $\mathbb{L}$ .*

*Proof.* Set  $G = A(L)$ . We want to prove that  $G$  is isometrically isomorphic to  $\mathbb{G}$ . Suppose that  $\varepsilon > 0$ , and  $n \in \mathbb{N}$ . Let  $\phi : \ell_n^\infty \rightarrow \ell_{n+1}^\infty$  and  $f : \ell_n^\infty \rightarrow G$  be linear isometries. We want to prove that there exists an isometric linear map  $\hat{f} : \ell_{n+1}^\infty \rightarrow G$  such that  $\|\hat{f} \circ \phi - f\| < \varepsilon$ . This will suffice in view of the characterization of the limit provided by Proposition 2.12. In view of Proposition 6.5, it is enough to find a facial quotient map  $Q : G \rightarrow \ell_{n+1}^\infty$  such that  $\|Q \circ f - \phi\| < \varepsilon$ . Fix  $\eta > 0$ . Choose standard bases  $e_1^n, \dots, e_n^n$  of  $\ell_n^\infty$  and  $e_1^{n+1}, \dots, e_{n+1}^{n+1}$  of  $\ell_{n+1}^\infty$  and  $a_1, \dots, a_n \in \mathbb{R}$  such that  $|a_1| + \dots + |a_n| \leq 1$  and  $\phi(e_i^n) = e_i^{n+1} + a_i^{n+1} e_{n+1}^{n+1}$ . For every  $i = 1, 2, \dots, n$  pick  $s_i \in \partial_e L$  such that  $s_i(f(e_i^n)) = 1$ . Since  $L$  is a nontrivial metrizable Lazar simplex with dense extreme boundary, one can find then  $s_{n+1} \in \partial_e L$  such that  $s_{n+1}$  does not belong to the absolutely convex hull of  $\{s_1, \dots, s_n\}$ , and

$$\left| s_{n+1}(f(e_i^n)) - \sum_{j=1}^n a_j s_j(f(e_i^n)) \right| < \eta$$

for  $i = 1, \dots, n$ . Let  $Q : G \rightarrow \ell_{n+1}^\infty$  be the map  $x \mapsto (s_1(x), \dots, s_{n+1}(x))$ . By [3, Proposition 2.3],  $Q$  is a quotient mapping. Observe that  $\|Q \circ f - \phi\| < \varepsilon$  for  $\eta$  small enough. For  $k = 1, 2, \dots, n+1$  define  $H_k$  to be  $\{\lambda s_k : \lambda \in [-1, 1]\}$ , and observe that  $H_k$  is a closed biface since  $s_k$  is an extreme point of  $L$ . Set now  $\hat{H}$  to be the convex hull of  $H_1, \dots, H_{n+1}$ . By [33, Proposition 4.6],  $\hat{H}$  is a closed biface of  $L$ . Since  $Q^\dagger \text{Ball}(\ell_n^1) = \hat{H}$ , we have that  $Q$  is a facial quotient mapping. This concludes the proof.  $\square$

**Proposition 6.9.** *Suppose that  $X$  is a separable Lindenstrauss space, and  $P : \mathbb{G} \rightarrow X$  is a contractive linear map. Then  $P$  belongs to the  $\text{Aut}(\mathbb{G})$ -orbit of  $\Omega_{\mathbb{G}}^X$  if and only if  $P$  is a nontrivial facial quotient.*

*Proof.* We have already observe that  $\Omega_{\mathbb{G}}^X$  is a nontrivial facial quotient. We prove the converse implication. Let  $H$  be the image of  $\text{Ball}(X^*)$  under the dual map  $P^\dagger$ . Suppose that  $\varepsilon > 0$ , and  $n \in \mathbb{N}$ . Let  $\phi : \ell_n^\infty \rightarrow \ell_{n+1}^\infty$  and  $f : \ell_n^\infty \rightarrow \mathbb{G}$  be linear isometries, and  $s : \ell_{n+1}^\infty \rightarrow X$  be a linear map such that  $P \circ f = s \circ \phi$ . In view of the characterization of the generic state from Subsection 5.3, it is enough to prove that there exists a linear isometry  $\hat{f} : \ell_{n+1}^\infty \rightarrow X$  such that  $P \circ \hat{f} = s$  and  $\|\hat{f} \circ \phi - f\| < \varepsilon$ . By Proposition 6.5, it is enough to prove that there exists a linear map  $Q : \mathbb{G} \rightarrow \ell_{n+1}^\infty$  such that  $Q \oplus P : \mathbb{G} \rightarrow \ell_{n+1}^\infty \oplus^\infty X$  is a facial quotient, and  $\|Q \circ f - \phi\| < \varepsilon$ . For this purpose one can proceed as in the proof of Proposition 6.8 and define  $s_1, \dots, s_{n+1}$ . Let then for  $k = 1, 2, \dots, n$ ,  $t_k \in \partial_e \mathbb{L} \setminus H$  such that  $|t_k(e_i) - s_k(e_i)| \leq \eta$  for every  $i = 1, 2, \dots, n$ . Define now  $H_k$  to be  $\{\lambda t_k : \lambda \in [-1, 1]\}$  for  $k = 1, 2, \dots, n+1$ , and  $\hat{H}$  to be convex hull of  $H_1, \dots, H_{n+1}$  and  $H$ . As in the proof of Proposition 6.8,  $\hat{H}$  is a closed biface. Let  $Q : \mathbb{G} \rightarrow \ell_{n+1}^\infty$  be the map  $x \mapsto (t_1(x), \dots, t_{n+1}(x))$ , and observe that it is a quotient mapping. The image of  $\text{Ball}(\ell_n^\infty \oplus^\infty X)$  under the dual map of  $P \oplus Q$  is  $\hat{H}$ . This shows that  $Q$  is a facial quotient. For  $\eta > 0$  small enough, one has that  $\|Q \circ f - \phi\| < \varepsilon$ , concluding the proof.  $\square$

The following corollary is an immediate consequence of Remark 5.7 and Proposition 6.9.

**Corollary 6.10.** *Any symmetric affine homeomorphism between closed proper bifaces of  $\mathbb{L}$  extends to a symmetric affine homeomorphism of  $\mathbb{L}$ .*

If  $L$  is a Lazar simplex, and  $Z \subset \partial_e L$  is a compact subset, then the absolutely convex hull  $H$  of  $Z$  is a closed biface of  $L$  such that  $\partial_e H = Z$  [33, Theorem 5.8]. By [33, Lemma 3.1] one can identify  $A_\sigma(H)$  with the space  $C_\sigma(Z)$  of continuous real-valued symmetric functions on  $Z$ .



**Corollary 6.11.** *A symmetric homeomorphism between proper compact subsets of  $\partial_e \mathbb{L}$  extends to a symmetric affine homeomorphism of  $\mathbb{L}$ .*

*Proof.* Suppose that, for  $i = 0, 1$ ,  $Z_i \subset \partial_e L$  is a proper compact subset,  $H_i$  is the absolutely convex hull of  $Z_i$ , and  $\varphi : Z_0 \rightarrow Z_1$  is a symmetric homeomorphism. Then  $\varphi$  induces an isometric isomorphism  $\alpha$  from  $C_\sigma(Z_1)$  to  $C_\sigma(Z_0)$ . Since as remarked above one can identify  $C_\sigma(Z_i)$  with  $A_\sigma(H_i)$ ,  $\alpha$  in turn induces a symmetric affine homeomorphism  $\widehat{\varphi}$  from  $H_0$  to  $H_1$  that extends  $\varphi$ . Applying Corollary 6.10 one can deduce that  $\widehat{\varphi}$  can be extended to a symmetric affine homeomorphism of  $\mathbb{L}$ .  $\square$

Suppose that  $X$  is a Lindenstrauss space,  $K = \text{Ball}(X^*)$  is the associated Lazar simplex, and  $H$  is a proper closed biface of  $K$ . Let  $N$  be the linear span of  $H$  inside  $X^*$  and  $e : X^* \rightarrow X^*$  be the corresponding  $L$ -projection. Then the range of  $I - e$  is the complementary (convex) cone  $N'$  of  $N$ ; see [2, Proposition 3.1]. The quotient mapping  $X^* \rightarrow X^*/N$  induces a linear isometry from  $N'$  onto  $X^*/N$  [3, Proposition 1.14]. The complementary biface of  $H$  is the intersection of  $N'$  with  $\text{Ball}(X^*)$ .

**Corollary 6.12.** *Suppose that  $H$  is a proper closed biface of  $\mathbb{L}$ . Endow the complementary biface  $H'$  with the  $w^*$ -topology induced by  $a \in \mathbb{G}$  such that  $a|_H \equiv 0$ . Then  $H'$  is affinely homeomorphic to  $\mathbb{G}$ .*

*Proof.* Suppose that  $H$  is a closed proper biface of  $\mathbb{L}$ . Consider  $J = \{f \in A_\sigma(\mathbb{L}) : f|_H = 0\}$  and set  $N := J^\perp$ . Observe that  $N$  coincides with the linear span of  $H$  inside  $\mathbb{G}^*$ . Let  $N'$  be the complementary cone of  $N$ , and  $H' = H \cap K$  be the complementary biface of  $H$ . By Proposition 6.9,  $J$  is isometrically isomorphic to  $\mathbb{G}$ , and  $\text{Ball}(J^*)$  is symmetrically affinely homeomorphic to  $\mathbb{L}$ . The inclusion  $J \subset \mathbb{G}$  induces by duality  $w^*$ -continuous linear map  $\varphi : \mathbb{G}^* \rightarrow J^*$ . We claim that the restriction of  $\varphi$  to  $N'$  is 1:1 and, in fact, isometric. Indeed, as observed in the proof of [3, Lemma 3.4(b)], one can identify  $\varphi$  with the quotient mapping  $\mathbb{G}^* \rightarrow \mathbb{G}^*/N$ . Therefore we have that  $H'$  with the topology described in the statement is symmetrically affinely homeomorphic to  $\text{Ball}(\mathbb{G}) = \text{Ball}(J^*) = \mathbb{L}$ .  $\square$

Theorem 1.2 and Theorem 5.1 now follow from the general results of Sections 2, 3, 4, 5 together with remarks above.

**6.2. Complex Banach spaces.** One can regard complex Banach spaces as structures in a suitable language  $\mathcal{L}$  similarly as real Banach spaces. In this section we will assume all the Banach spaces to be complex and separable. The finite-dimensional injective complex Banach spaces are precisely those of the form  $\ell_n^\infty$  (finite  $\infty$ -sum of  $n$  copies of  $\mathbb{C}$ ) for some  $n \in \mathbb{N}$ . The Fraïssé limit of the class of finite-dimensional complex Banach spaces is the complex Gurarij space  $\mathbb{G}$ . The analogue of Fact 6.1 for complex Banach spaces holds due to results of Hustad [54], Olsen [94], and Nielsen-Olsen [90]. We call a complex Banach space satisfying the (complex analogs of) any of the equivalent properties of Fact 6.1 a complex Lindenstrauss space.

**Definition 6.13.** *A compact circled convex set is a compact subset  $K$  of a complex locally convex topological vector space such that  $\lambda x + \mu y \in K$  whenever  $x, y \in K$  and  $\lambda, \mu \in \mathbb{C}$  are such that  $|\lambda| + |\mu| \leq 1$ .*

Let  $K$  be a compact circled convex set and  $\xi \in \mathbb{T}$ . We denote by  $\sigma_\xi : K \rightarrow K$  the map  $p \mapsto \xi p$ . A complex-valued function  $f$  on  $K$  is called  $\mathbb{T}$ -invariant if  $f \circ \sigma_\xi = f$  for every  $\xi \in \mathbb{T}$ , and  $\mathbb{T}$ -homogeneous if  $f \circ \sigma_\xi = \xi f$  for every  $\xi \in \mathbb{T}$ . Similar definitions apply to complex Borel measures on  $K$ . The map  $f \mapsto \text{inv}_\mathbb{T} f = \int (f \circ \sigma_\xi) d\xi$  is a norm 1 projection from  $C(K)$  onto the space of continuous  $\mathbb{T}$ -invariant functions, while the map  $f \mapsto \text{hom}_\mathbb{T} f = \int \xi (f \circ \sigma_\xi) d\xi$  is a norm 1 projection onto the space of continuous  $\mathbb{T}$ -homogeneous functions. The adjoints of these projections give  $w^*$ -continuous projections  $\mu \mapsto \text{inv}_\mathbb{T} \mu$  and  $\mu \mapsto \text{hom}_\mathbb{T} \mu$  of the space  $M(K)$  of complex Borel probability measures onto the spaces of  $\mathbb{T}$ -invariant and  $\mathbb{T}$ -homogeneous measures, respectively. A continuous map  $\phi : K_0 \rightarrow K_1$  is  $\mathbb{T}$ -homogeneous if  $\phi \circ \sigma_\xi = \sigma_\xi \circ \phi$  for every  $\xi \in \mathbb{T}$ . Let  $A_\mathbb{T}(K) \subset C(K)$  be the space of continuous  $\mathbb{T}$ -homogeneous complex-valued functions on  $K$ . It follows from the geometric Hahn-Banach theorem for complex Banach spaces that the map sending  $p \in K$  to the corresponding evaluation functional in  $A_\mathbb{T}(K)^*$  is a  $\mathbb{T}$ -homogeneous affine homeomorphism of  $K$  onto the unit ball of  $A_\mathbb{T}(K)^*$ . Conversely, one can identify a complex Banach space  $X$  with  $A_\mathbb{T}(K)$  where  $K$  is the unit ball of the dual space of  $X$  endowed with the  $w^*$ -topology. The function  $K \mapsto A_\mathbb{T}(K)$  is a contravariant equivalence of categories from the category of compact circled convex sets and  $\mathbb{T}$ -homogeneous affine continuous functions to the category of complex Banach spaces and complex-linear maps of norm at most 1. In the following we assume all compact convex circled sets to be metrizable, and all the Banach spaces to be separable.

**Definition 6.14.** *An Effros simplex is the unit ball of the dual space of a complex Lindenstrauss space.*

Effros characterized in [34] what we call Effros simplices: a compact circled convex set  $K$  is an Effros simplex if and only if given boundary probability measures  $\mu_1, \mu_2$  of  $K$  with the same barycenter one has that  $\text{hom}_{\mathbb{T}}\mu_1 = \text{hom}_{\mathbb{T}}\mu_2$ .

All the definitions about simplices carry over with no change from the real to the complex setting, as well as the notions of collinear elements,  $L$ -ideals, and  $M$ -ideals. The notion of cone of a complex vector space is defined as in the real case: a subset  $C$  of a complex vector space is a cone if  $\lambda x \in C$  for any  $x \in C$  and  $\lambda \geq 0$ . A cone  $C$  in a dual Banach space  $X^*$  is *hereditary* if whenever  $p, q \in X^*$  are collinear and  $p + q \in C$  then  $p, q \in C$ .

**Lemma 6.15.** *Suppose that  $X$  is a complex Lindenstrauss space and that  $J$  is a subspace of  $X^*$ . The following assertions are equivalent.*

- (1)  $W$  is an  $L$ -ideal;
- (2)  $\|x + y\| = \|x\| + \|y\|$  for any  $x \in J$  and  $y \in J'$  (the complementary cone of  $J$ );
- (3)  $W$  is hereditary.

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. The implication (3) $\Rightarrow$ (2) is [76, Lemma 1]—after observing that a Banach space is an  $E(3)$  space if and only if it is a Lindenstrauss space—while the implication (2) $\Rightarrow$ (1) is [77, Theorem 5.5].  $\square$

**Definition 6.16.** *A subset  $H$  of a compact convex circled set  $K$  is a circled face if it is circled convex, it contains  $\|x\|^{-1}x$  whenever  $x \in H$  is nonzero, and if  $x, y \in K$  are codirectional and  $x + y \in H$  then  $x, y \in H$ .*

It follows from Lemma 6.15 and [41, Proposition 2.1] that if  $X$  is a Lindenstrauss space and  $K$  is the Effros simplex  $\text{Ball}(X^*)$ , then the closed circled faces of  $K$  are precisely the sets of the form  $J \cap K$  where  $J$  is a  $w^*$ -closed  $L$ -ideal of  $X^*$ . Indeed if  $H$  is a closed circled face of  $K$ , then the linear span  $J$  of  $K$  is a  $w^*$ -closed  $L$ -ideal of  $X^*$  such that  $H = J \cap K$ . Furthermore if  $A$  is a compact subset of  $\partial_e K$  then the closure of the circled convex hull of  $A$  is a circled face of  $X^*$ . The following is the natural complex analog of Proposition 6.17, and can be proved with similar methods by replacing [74, Theorem 2.2] with [94, Theorem 4.2].

**Proposition 6.17.** *Suppose that  $P : Z \rightarrow X$  is a quotient mapping between complex Lindenstrauss spaces. If  $H$  is the image of  $\text{Ball}(X^*)$  under  $P^\dagger$  and  $N$  is the kernel of  $P$ , then the following statements are equivalent:*

- (1)  $N$  is an  $M$ -ideal of  $X$ ;
- (2) whenever  $\varepsilon > 0$ ,  $E \subset F$  are finite-dimensional Banach spaces,  $g : F \rightarrow X$  is a linear contraction, and  $f : E \rightarrow Z$  is a linear isometry such that  $P \circ f = g|_E$ , then there exists a linear contraction  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\| \leq \varepsilon$ ;
- (3) whenever  $\varepsilon > 0$ ,  $A$  is a separable Banach space with the metric approximation property,  $E \subset A$  is a finite-dimensional subspace, and  $f : E \rightarrow Z$  and  $g : A \rightarrow X$  are linear contractions such that  $\|P \circ f - g|_E\| < \varepsilon$ , then there exists a linear contraction  $\hat{g} : A \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\| < 6\varepsilon$ ;
- (4) for any subspace  $E$  of  $Z$  and  $\varepsilon \geq 0$  one has that  $\|P(x)\| \geq (1 - \varepsilon)\|x\|$  for any  $x \in E$  if and only if there exists a linear contraction  $\eta : X \rightarrow Z$  such that  $P \circ \eta$  is the identity map of  $X$  and  $\|\eta \circ P|_E - \text{id}_E\| \leq \varepsilon$ .
- (5) for any  $\varepsilon > 0$ ,  $y, u \in Z$  such that  $\|y\| = \|u\| = 1$ , and  $u \in N$ , there exists  $v \in Z$  such that  $\|P(v)\| \leq \varepsilon$  and  $\|v - y + \xi u\| \leq 1 + \varepsilon$  for every  $\xi \in \mathbb{T}$ ;
- (6)  $H$  is a circled face of  $K$ .

**Remark 6.18.** *Similarly as for real Banach spaces, the equivalence of (1)–(3) in Proposition 6.17 holds without the assumption that  $Z, X$  are Lindenstrauss spaces; see Remark 6.6. Furthermore if  $H$  is a closed circled face of a metrizable Effros simplex  $K$ , then the restriction mapping  $A_{\mathbb{T}}(K) \rightarrow A_{\mathbb{T}}(H)$ ,  $f \mapsto f|_H$  is automatically a complete quotient mapping by [94, Theorem 4.2].*

As in the real case, we define a quotient mapping  $P : Z \rightarrow X$  to be *facial quotient* if it satisfies any of the equivalent conditions of Proposition 6.17. As in the real case, one can deduce the complex analog of Theorem 1.2 and Theorem 5.1 from Proposition 6.17 and the general results from Sections 2, 3, 4, 5.

**6.3. Function systems.** A *function system* is an ordered real vector space  $V$  endowed with a distinguished element  $e$  that is an *Archimedean order unit* [1, Chapter 2]. This means that, for every  $v \in V$ ,

- there exists  $n \in \mathbb{N}$  such that  $-ne \leq v \leq ne$ , and
- if, for every  $k \in \mathbb{N}$ ,  $kv \leq e$ , then  $v \leq 0$ .

An function system  $V$  is naturally endowed with a norm defined by

$$\|v\| = \inf \{r \in \mathbb{R}_+ : -re \leq v \leq re\}.$$

We will always assume such a norm to be complete. A state on  $V$  is a linear function  $s$  that is *positive* and *unital*. This means that  $s$  maps positive elements of  $V$  to positive real numbers, and maps the order unit of  $V$  to 1. The space  $S(V)$  of states of  $V$  is a  $w^*$ -compact convex subset of the dual  $V^*$  of  $V$ . A unital linear functional on  $V$  is positive if and only if it is contractive, and  $v \in V$  is positive if and only if  $s(v)$  is positive for every state  $s$  of  $V$ . Hence in a function system one can reconstruct the order from the norm and the order unit. Two function system  $V$  and  $W$  are order isomorphic if there exists a surjective unital linear isometry from  $V$  to  $W$ .

If  $K$  is a compact convex set, then the space  $A(K)$  of real-valued continuous affine functions on  $K$  is a function system with its usual order structure, maximum norm, and the function constantly equal to 1 as order unit. Kadison's representation theorem asserts that the map from  $V$  to  $A(S(V))$  mapping  $v$  to the evaluation function at  $v$  is a surjective unital linear isometry [1, Theorem II.1.8]; see also [57, 58]. Furthermore the map  $K \mapsto A(K)$  is a contravariant isomorphism from the category of compact convex sets and continuous affine maps to the category of function systems and unital contractive linear maps. Using this observation one can reformulate statements about compact convex sets into statements about function systems, and vice versa. Considering *complex* function systems rather than real function systems does not yield any substantial difference. Indeed, any complex function system is the complexification of a real function system, and any complex-linear unital map between complex function systems is the complexification of a real-linear unital linear map of the same norm.

We regard function systems as structures in the language of Banach spaces with an additional constant symbol for the order unit. Basic tuples in this context are linearly independent tuples whose first element is the order unit. We consider the collection  $\mathcal{I}$  of injective objects consisting of the spaces  $\ell_n^\infty$  for  $n \in \mathbb{N}$  with the  $n$ -tuple constantly equal to 1 as order unit. The following lemma can be proved as [36, Theorem 3.5] using [1, Proposition II.1.14]; see also Lemma 8.4.

**Lemma 6.19.** *Suppose that  $V, W$  are function systems, and  $f : V \rightarrow W$  is a unital linear map such that  $\|f\| \leq 1 + \delta$ . If  $W$  is injective, then there exists a unital positive linear map  $g : V \rightarrow W$  such that  $\|f - g\| \leq 2\delta$ . If  $W$  is arbitrary and  $V$  is  $n$ -dimensional, then there exists a unital positive linear map  $g : V \rightarrow W$  such that  $\|f - g\| \leq 2n\delta$ .*

It follows from Lemma 6.19 and the discussion above that the all the conditions of Section 2 are met with  $\varpi(\delta) = 2\delta$ . The following statement collects together classical results from the theory of compact convex sets; see [1, 55].

**Fact 6.20.** Suppose that  $V$  is a separable function system. The following conditions are equivalent:

- (1)  $V$  is a Lindenstrauss space;
- (2)  $V$  is approximately injective as in Definition 3.2,
- (3)  $V$  is an  $\mathcal{I}$ -structure as in Definition 3.6,
- (4)  $V$  is a rigid  $\mathcal{I}$ -structure as in Definition 3.6,
- (5)  $V$  is the direct limit of copies of  $\ell_n^\infty$  for  $n \in \mathbb{N}$  with unital linear isometries as connective maps,
- (6) the state space of  $V$  is a Choquet simplex.

We call a function system satisfying the equivalent conditions of Fact 6.20 above a *simplex space*. All the function systems below are assumed to be separable, and all the compact convex sets are assumed to be metrizable. One can conclude from the general results of Section 7 that finite-dimensional function systems form a Fraïssé class. Let us denote by  $A(\mathbb{P})$  the corresponding limit, and by  $\mathbb{P}$  the state space of  $A(\mathbb{P})$ . We will show below that  $\mathbb{P}$  is the unique metrizable Choquet simplex with dense extreme boundary.

Suppose that  $A(K)$  and  $A(F)$  are function systems with state spaces  $K$  and  $F$  respectively. We let  $S(A(K), A(F))$  be the space of unital positive linear maps  $\phi : A(K) \rightarrow A(F)$  endowed with the topology of pointwise convergence. Let also  $A(F, K)$  be the space of continuous affine functions  $f : F \rightarrow K$  endowed with the compact open topology and its natural convex structure. The assignment  $\phi \mapsto \phi^\dagger$  where  $\phi^\dagger s = s \circ \phi$  is a homeomorphism from  $S(A(K), A(F))$  onto  $A(F, K)$ .

We say that a function system  $A$  has the metric approximation property if it has such a property as a Banach space. In view of Lemma 6.19 this is equivalent to the assertion that the identity map of  $A$  is the pointwise limit of finite rank unital positive linear maps.

**Proposition 6.21.** *Suppose that  $V, W$  are separable simplex spaces, and  $P : V \rightarrow W$  is unital quotient mapping. Let  $K$  be the state space of  $V$ ,  $H$  be the image under  $P^\dagger$  of the state space of  $W$ , and  $N$  be the kernel of  $P$ . The following statements are equivalent:*

- (1)  $N$  is an  $M$ -ideal of  $X$

- (2) whenever  $\varepsilon > 0$ ,  $E_0 \subset E_1$  are finite-dimensional function systems,  $g : E_1 \rightarrow W$  is a unital positive linear map, and  $f : E_0 \rightarrow V$  is a unital linear isometry such that  $P \circ f = g|_{E_0}$ , then there exists a unital positive linear map  $\widehat{g} : E_1 \rightarrow V$  such that  $P \circ \widehat{g} = g$  and  $\|\widehat{g}|_{E_0} - f\| \leq \varepsilon$ ;
- (3) whenever  $\varepsilon > 0$ ,  $A$  is a separable function system with the metric approximation property,  $E$  is a finite-dimensional subspace of  $A$ ,  $f : E \rightarrow V$  and  $g : A \rightarrow W$  are unital positive linear maps with  $\|P \circ f - g|_E\| < \varepsilon$ , then there exists a unital positive linear map  $\widehat{g} : A \rightarrow V$  such that  $P \circ \widehat{g} = g$  and  $\|\widehat{g}|_E - f\| < 3\varepsilon$ ;
- (4) if  $\varepsilon \geq 0$  and  $E \subset V$  is a subsystem such that  $\|x\| \leq (1 + \varepsilon) \|P(x)\|$  for every  $x \in E$ , then there exists a linear contraction  $\eta : W \rightarrow V$  such that  $P \circ \eta$  is the identity map of  $W$  and  $\|\eta \circ P|_E - \text{id}_E\| \leq 2\varepsilon$ .
- (5) for any  $\varepsilon > 0$ ,  $u \in N$  such that  $\|u\| = 1$ , there exists an  $v$  of  $V$  such that  $0 \leq v \leq 1$ ,  $\|P(v)\| \leq \varepsilon$ , and  $v \geq u - \varepsilon$ ;
- (6)  $H$  is closed face of  $K$ .

*Proof.* In the proof we identify  $V$  with  $A(K)$  and  $W$  with  $A(H)$ . Under these identifications  $P$  is just the restriction mapping  $A(K) \rightarrow A(H)$ ,  $f \mapsto f|_H$ . Observe that in a Choquet simplex every closed face is a split face [1, Theorem II.6.22]. The equivalence of (6) and (1) thus follows from [3, Corollary 5.9, Proposition 5.10]. The implication (6) $\Rightarrow$ (3) can be proved as [20, Theorem 2.6] using [18, Proposition 2.2] instead of [20, Lemma 2.1]. The implication (2) $\Rightarrow$ (5) can be proved as (2) $\Rightarrow$ (1). We prove the other nontrivial implications below.

(5) $\Rightarrow$ (6) Suppose that  $q_0, q_1 \in K$  and  $p \in H$  are such that  $(q_0 + q_1)/2 = p$ . We want to prove that  $q_0 \in H$ . Suppose that  $u$  is an element of  $N$  of norm 1 and  $\varepsilon > 0$ . By assumption there exists an element  $v$  of  $V$  such that  $0 \leq v \leq 1$ ,  $\|P(v)\| \leq \varepsilon/2$ , and  $v \geq u - \varepsilon/2$ . Then we have

$$q_0(v) \leq \varepsilon - q_1(v) \leq \varepsilon/2$$

and

$$q_0(u) = q_0(v) + q_0(u - v) \leq \varepsilon.$$

(6) $\Rightarrow$ (5) Consider the function

$$\varphi : k \mapsto \{t \in \mathbb{R} : \max\{k(u), 0\} \leq t \leq 1\}$$

and observe that it satisfies the hypothesis of [71, Corollary 3.4], and  $0 \in \varphi(k)$  for every  $k \in H$ . It follows that there exists  $v \in N$  such that  $k(v) \in \varphi(k)$  for every  $k \in K$ .

(6) $\Rightarrow$ (4): Suppose that  $E \subset V$  is a subsystem such that  $\|P(x)\| \geq (1 - \varepsilon) \|x\|$  for every  $x \in E$ . Let  $k$  be an element of  $K$ . We observe that there exists  $h \in H$  such that  $\|(k - h)|_E\| \leq 2\varepsilon$ . Define the map  $h_0 : P[E] \rightarrow \mathbb{R}$  by  $h_0(P(e)) = k(e)$ . Observe that by assumption  $h_0$  is a well-defined unital linear functional such that  $\|h_0\| \leq 1 + \varepsilon$ . Therefore by Lemma 6.19 there exists a state  $h_1$  of  $W$  such that  $\|h_0 - h_1\| \leq 2\varepsilon$ . One can then define  $h := h_1 \circ P \in H$  and observe that  $\|(h - k)|_E\| \leq 2\varepsilon$ . Consider the function defined by

$$\varphi : k \mapsto \{h \in H : \|(k - h)|_E\| \leq 2\varepsilon\}.$$

Observe that  $\varphi$  satisfies the assumptions of [71, Corollary 3.4]. Hence there exists a continuous affine function  $Q : K \rightarrow H$  such that  $Q|_H$  is the identity map of  $H$  and  $\|(Q(k) - k)|_E\| \leq \varepsilon$  for every  $k \in K$ . One can thus define  $\eta : A(H) \rightarrow A(K)$  by  $\eta \mapsto \eta \circ Q$ .

(2) $\Rightarrow$ (1) Fix  $y^{(1)}, y^{(2)}, y^{(3)} \in \text{Ball}(N)$ ,  $x \in \text{Ball}(X)$ , and  $\varepsilon > 0$ . By the equivalence (i) $\Leftrightarrow$ (iv) in [53, Theorem 2.2] it is enough to prove that there exists  $y \in \text{Ball}(N)$  such that  $\|x + y^{(\ell)} - y\| \leq 1 + \varepsilon$  for every  $\ell \in \{1, 2, 3\}$ . Let  $E$  be the linear span of  $\{y^{(1)}, y^{(2)}, y^{(3)}, x, 1\}$  inside  $Z$ . Consider the function system obtained from  $F \oplus \mathbb{R}$  and the collection of linear functions  $(z, \alpha) \mapsto s(z) + t\alpha$  where  $t \in [-1, 1]$  and  $s$  is a state of  $F$  such that  $|s(x + y^{(\ell)}) - t| \leq 1$  for every  $\ell \in \{1, 2, 3\}$ . Define also the linear map  $g : F \rightarrow X$  by  $(z, \alpha) \mapsto P(z)$ . Observe that the inclusion  $E \subset F$  is a unital linear isometry while  $g$  is a unital positive linear map such that  $g|_E = P$ . By assumption there exists a unital positive linear map  $\widehat{g} : F \rightarrow Z$  such that  $P \circ \widehat{g} = g$  and  $\|\widehat{g}|_E - \iota_E\| \leq \varepsilon$ , where  $\iota_E : E \rightarrow Z$  is the inclusion map. Setting  $y := \widehat{g}(0, 1)$  concludes the proof.  $\square$

**Remark 6.22.** As in Proposition 6.21 and Proposition 6.5, the equivalence of (1)–(3) in Proposition 6.21 holds for arbitrary separable function systems  $V, W$ . If  $F$  is a closed face of a metrizable Choquet simplex  $K$ , then the function  $A(K) \rightarrow A(F)$ ,  $f \mapsto f|_F$  is automatically a quotient mapping by [71].

We call a unital quotient mapping  $P : A(K) \rightarrow A(F)$  between simplex spaces satisfying the equivalent conditions of Proposition 6.21 *unital facial quotient*. A unital facial quotient  $P : A(K) \rightarrow A(F)$  is *nontrivial* if it is not an order isomorphism or, equivalently,  $P^\dagger F$  is a proper face of  $K$ .

Suppose that  $F$  is a Choquet simplex. It follows from Proposition 6.21 that the universal positive linear map  $\Omega_{A(\mathbb{P})}^{A(F)} : A(\mathbb{P}) \rightarrow A(F)$  from Subsection 5.3 is a unital facial quotient mapping. In particular when  $K$  is

the trivial simplex one obtains an extreme point  $\Omega_{A(\mathbb{P})}^{\mathbb{R}} : A(\mathbb{P}) \rightarrow \mathbb{R}$  of the state space  $\mathbb{P}$  of  $A(\mathbb{P})$ . Since  $\Omega_{A(\mathbb{P})}^{\mathbb{R}}$  has a dense  $G_\delta$  orbit in  $\mathbb{P}$ , we conclude that  $\mathbb{P}$  has dense extreme boundary. Conversely assuming that  $S$  is a metrizable Choquet simplex with dense extreme boundary, one can prove that  $A(S)$  is unital isometrically isomorphic to  $A(\mathbb{P})$  arguing as in Proposition 6.8. Thus  $\mathbb{P}$  is the unique metrizable Choquet simplex with dense extreme boundary.

Suppose now that  $F$  is a Choquet simplex, and  $\Omega_{A(\mathbb{P})}^{A(F)} : A(\mathbb{P}) \rightarrow A(F)$  is the generic unital positive linear map obtained from the general results of Subsection 5.3. The proof of Proposition 6.9 can be adapted in a straightforward way to show that a unital quotient mapping  $P : A(\mathbb{P}) \rightarrow A(F)$  is a unital facial quotient if and only if it belongs to the  $\text{Aut}(A(\mathbb{P}))$ -orbit of  $\Omega_{A(\mathbb{P})}^{A(F)}$ , if and only if the image of  $F$  under  $P^\dagger$  is a closed proper face of  $\mathbb{P}$ . It follows that any metrizable Choquet simplex is affinely homeomorphic to a closed proper face of  $\mathbb{P}$  [79, Theorem 2.5], and any affine homeomorphism between proper closed faces of  $\mathbb{P}$  extends to an affine homeomorphism of  $\mathbb{P}$ . Furthermore if  $F$  is a closed proper face of  $\mathbb{P}$ , then  $\{f \in A(\mathbb{P}) : f \text{ is constant on } F\}$  is a function system order isomorphic to  $A(\mathbb{P})$ .

Suppose that  $K, K_0$  are Choquet simplices,  $\varphi : K \rightarrow K_0$  is a surjective continuous affine map,  $F$  is a proper closed face of  $K$ , and  $F'$  is the complementary face of  $F$ ; see [1, Section 6]. It follows from Edwards' separation theorem [1, Theorem II.3.10] and [1, Proposition II.6.5] that  $F'$  is the set of points  $s \in K$  such that for any  $\varepsilon > 0$  there exists  $h \in A(K)$  such that  $0 \leq h \leq 1$ ,  $h|_F$  is constantly equal to 1, and  $h(s) < \varepsilon$ . It follows that the image of  $F$  under  $\varphi$  is a closed face  $F_0$  of  $K_0$ , and the image of  $F'$  under  $\varphi$  is the complementary face of  $F_0$ .

**Proposition 6.23** ([79, Corollary 2.4]). *Suppose that  $F_0, F_1$  are proper closed faces of  $\mathbb{P}$ . Endow the complementary face  $F'_0$  of  $F_0$  with the  $w^*$ -topology induced by the elements  $a$  of  $A(\mathbb{P})$  such that  $a|_{F_0}$  is constant, and similarly for  $F'_1$ . Then  $F'_0$  and  $F'_1$  are affinely homeomorphic.*

*Proof.* Suppose that  $F$  is a proper closed face of  $\mathbb{P}$  and  $F'$  is the complementary face of  $F$ . Consider  $W = \{f \in A(\mathbb{P}) : f \text{ is constant on } F\}$  and observe that  $W$  is a function system. Let  $K$  be the state space of  $W$ . As observed before,  $K$  is affinely homeomorphic to  $\mathbb{P}$ . Denote by  $\varphi : \mathbb{P} \rightarrow K$  the surjective continuous affine map obtained from the inclusion  $A(K) \subset A(\mathbb{P})$  by duality. The image of  $F$  is a single point  $s_F$  of  $K$ . Since  $F$  is a face,  $s_F$  is an extreme point of  $K$ . The map  $\varphi$  is 1:1 on the complementary face  $F'$  of  $F$  by [1, Corollary II.6.17]. It follows from the remarks above that the image of  $F'$  under  $\varphi$  is the complementary face of  $\{s_F\}$  in  $K$ . The  $w^*$ -topology on  $F'$  induced by the elements of  $W$  makes the restriction of  $\varphi$  to  $F'$  a homeomorphism. The conclusion now follows from the fact that  $K$  is affinely homeomorphic to  $\mathbb{P}$  and that  $\text{Aut}(\mathbb{P})$  acts transitively on the extreme points of  $\mathbb{P}$ .  $\square$

Applying the criterion from Corollary 5.9 one can see that the canonical continuous action of  $\text{Aut}(\mathbb{P})$  on  $\mathbb{P}$  is minimal, recovering a result of Glasner from [47, Theorem 5.2].

**Proposition 6.24.** *The canonical action  $\text{Aut}(\mathbb{P}) \curvearrowright \mathbb{P}$  is minimal.*

*Proof.* In view of Proposition 5.8 it is enough to prove that for any  $\varepsilon > 0$  and  $d \in \mathbb{N}$  there exists  $m \in \mathbb{N}$  such that for any  $s \in S(\ell_d^\infty)$  and  $t \in S(\ell_m^\infty)$  there exists a unital linear isometry  $\phi : \ell_d^\infty \rightarrow \ell_m^\infty$  such that  $\|t \circ \phi - s\| < \varepsilon$ . Set  $\eta = \frac{\varepsilon}{2d}$  and  $m \in \mathbb{N}$  be such that  $m \geq 1/\eta + d$ . Suppose that  $s \in S(\ell_d^\infty)$  and  $t \in S(\ell_m^\infty)$ . Then  $s = (s_1, \dots, s_d)$  can be seen as a stochastic vector of length  $d$ , and  $t = (t_1, \dots, t_m)$  can be seen as a stochastic vector of length  $m$ . Let  $A \subset \{1, 2, \dots, m\}$  be the set of  $k$  such that  $t_k \geq \eta$ . Observe that  $|A| \leq 1/\eta$ . We can assume that  $A = \{1, 2, \dots, \ell\}$  for some  $\ell \leq 1/\eta$ . Define the map  $\phi : \ell_d^\infty \rightarrow \ell_m^\infty$  by  $x = (x_1, \dots, x_d) \mapsto (s(x), \dots, s(x), x_1, \dots, x_d)$ . Observe that  $\phi$  is indeed a unital linear isometry. Furthermore we have that, for  $x \in \ell_d^\infty$  such that  $\|x\| \leq 1$ ,

$$\begin{aligned} |(t \circ \phi)(x) - s(x)| &= |s(x)(t_1 + \dots + t_{m-d}) + x_1 t_{m-d+1} + \dots + x_d t_m - s(x)| \\ &= |x_1 t_{m-d+1} + \dots + x_d t_m - s(x)(t_{m-d+1} + \dots + t_m)| \leq 2d\eta \leq \varepsilon. \end{aligned}$$

This concludes the proof.  $\square$

As for Banach spaces, one can conclude from uniqueness of the Fraïssé limit, the characterization of the Fraïssé limit, and [8, Proposition 13.6] that the following facts, already proved implicitly in [49], hold: the first order theory of  $A(\mathbb{P})$  has a unique separable model and it admits elimination of quantifiers; the group  $\text{Aut}(\mathbb{P})$  of affine homeomorphisms of  $\mathbb{P}$  is Roelcke precompact;  $A(\mathbb{P})$  is the unique existentially closed separable function system; the theory of  $A(\mathbb{P})$  is the model completion of the theory of function systems; a compact convex set  $K$  is a Choquet simplex if and only if  $A(K)$  is a positively existentially closed function system.

**6.4.  $p$ -multinormed spaces.** Fix  $p \in [1, +\infty]$ . Consider the space  $B(\ell^p)$  of bounded linear operators on  $\ell^p$ , and let  $\mathcal{K}^p \subset B(\ell^p)$  be the space of compact operators. Observe that if  $X$  is a complex vector space, then the algebraic tensor product  $\ell^p \otimes X$  has a natural left  $\mathcal{K}^p$ -module structure. A  $p$ -multinormed space is a complex vector space such that  $\ell^p \otimes X$  is endowed with a norm such that  $\|\alpha x\| \leq \|\alpha\| \|x\|$  for  $\alpha \in \mathcal{K}^p$  and  $x \in \ell^p \otimes X$ , where  $\|\alpha\|$  denotes the norm of  $\alpha$  regarded as an element of  $B(\ell^p)$ . A linear map  $\phi : X \rightarrow Y$  between  $p$ -multinormed spaces is multicontractive if  $\text{id}_{\mathcal{K}^p} \otimes \phi$  is contractive, and multi-isometric if  $\text{id}_{\mathcal{K}^p} \otimes \phi$  is isometric. If  $X, Y$  are  $p$ -multinormed spaces, then the  $\infty$ -sum  $X \oplus^\infty Y$  is defined by identifying isometrically  $\ell^p \otimes (X \oplus Y)$  with the  $\infty$ -sum of  $\ell^p \otimes X$  and  $\ell^p \otimes Y$ . One can similarly define the  $\infty$ -sum of an arbitrary collection of  $p$ -multinormed spaces.

Multinormed spaces have been introduced and studied in [24, 23, 25]. The generalization to  $p$ -multinormed spaces for arbitrary  $p \in [1, +\infty]$  has been studied in the recent work of Dales, Laustsen, Oikhberg, and Troitsky [27, 26]. Multinormed spaces correspond to the case  $p = +\infty$ . If  $E$  is a Banach space, we denote by  $\max^p(E)$  the largest compatible  $p$ -multinorm structure on  $E$ . In the following we will always assume  $p \in (1, +\infty)$ . It has been recently shown by Oikhberg [92] that, if  $q$  is the conjugate exponent of  $p$  and  $(X, \mu)$  is a measure space, then  $\max^p(L^q(X, \mu))$  is an injective  $p$ -multinormed space. Furthermore any  $p$ -multinormed space embeds multi-isometrically into the  $\infty$ -sum of  $p$ -multinormed spaces of the form  $\max^p(\ell_n^q)$ .

Let  $\mathcal{L}$  be the language containing binary function symbols  $f_{\alpha, \beta}$  for any  $\alpha, \beta \in \mathcal{K}_0^p(\mathbb{Q}(i))$  such that  $\|\alpha\| + \|\beta\| \leq 1$ . Here  $\mathbb{Q}(i)$  is the field of Gauss rationals, while  $\mathcal{K}_0^p(\mathbb{Q}(i))$  denotes the space of operators whose representative matrices with respect to the canonical basis of  $\ell^p$  have coefficients that belong to  $\mathbb{Q}(i)$ , and are all zero but finitely many. A  $p$ -multinormed space  $X$  can be regarded as an  $\mathcal{L}$ -structure supported by the unit ball of  $\ell^p \otimes X$ , where  $f_{\alpha, \beta}$  is interpreted as the function  $(x, y) \mapsto \alpha x + \beta y$ . A morphism in this context is a linear multicontraction, and an embedding is a linear multi-isometry.

We can then consider the category  $\mathcal{A}$  of  $p$ -multinormed spaces and multicontractive maps, and the collection  $\mathcal{I} \subset \mathcal{A}$  of  $p$ -multinormed spaces that are a finite  $\infty$ -sum of copies of  $\max^p(\ell_n^q)$ . If  $f : X \rightarrow Y$  is a linear multicontraction, then  $I(f) \leq \delta < 1$  as in Definition 2.3 if and only if  $\|(\text{id}_{\mathcal{K}^p} \otimes f)(x)\| \geq \|x\| - \delta$  for any  $x \in \mathcal{K}^p \otimes X$  of norm at most 2, which happens if and only if  $T$  is injective and  $\|\text{id}_{\mathcal{K}^p} \otimes T^{-1}\| \leq 1 + \delta$ . We stipulate that a finite tuple  $\bar{a}$  in a  $p$ -multinormed space is a basic tuple if it is linearly independent. An argument similar to the small perturbations lemma [98, Lemma 2.13.2] shows that the conditions from Subsection 2.2 are satisfied.

We can conclude that finite-dimensional  $p$ -multinormed spaces form a Fraïssé class. We call the corresponding limit  $\mathbb{GM}^p$  the *Gurarij  $p$ -multinormed space*. It has been proved by Oikhberg [92] that for every  $\varepsilon > 0$  and  $n \in \mathbb{N}$  there exists  $k \in \mathbb{N}$  (depending only on  $n$  and  $\varepsilon$ ) with the following property: for any  $n$ -dimensional  $p$ -multinormed spaces  $E, F$  and linear map  $\phi : E \rightarrow F$ , one has that  $\|\text{id}_{\mathcal{K}^p} \otimes \phi\| \leq (1 + \varepsilon) \|\text{id}_{M_k^p} \otimes \phi\|$ . Here we regard  $M_k^p \subset \mathcal{K}^p$  as the subspace of  $x \in \mathcal{K}^p$  such that  $p_k x = x p_k = x$ , where  $p_k$  is the projection onto the span of the first  $k$  vectors of the canonical basis of  $\ell^p$ . Therefore the characterizing property of  $\mathbb{GM}^p$  given by Proposition 2.12 is elementary, that is, it can be expressed by formulas in the logic for metric structures. Hence  $\mathbb{GM}^p$  is the unique separable model of its first order theory. As for the Gurarij space, one can also observe that by [8, Proposition 13.6] the theory of  $\mathbb{GM}^p$  admits elimination of quantifiers. It follows from this and [10, Theorem 2.4] that the Polish group of surjective linear multi-isometries of  $\mathbb{GM}^p$  endowed with the topology of pointwise convergence is Roelcke precompact; see [10, Definition 2.2]. Since every  $p$ -multinormed space embeds into a model of the theory of  $\mathbb{GM}^p$ , one can conclude that the theory of  $\mathbb{GM}^p$  is the model completion of the theory of  $p$ -multinormed spaces. Finally one can observe that for a separable  $p$ -multinormed space  $X$ , the following assertions are equivalent:

- $X$  is approximately injective as in Definition 3.2;
- $X$  is multi-isometric to the range of  $p$ -multicontractive linear projections on  $\mathbb{GM}^p$ ;
- the identity map of  $X$  is the pointwise limit of multicontractive maps that factor through finite  $\infty$ -sums of copies of  $\max^p(\ell_n^p)$  for  $n \in \mathbb{N}$ ;
- $X$  is positively existentially closed in the class of  $p$ -multinormed spaces.

**6.5. Operator sequence spaces.** An *operator sequence space* is a 2-multinormed space which is moreover 2-convex, in the sense that it satisfies

$$\left\| \sum_{i=1}^n e_i \otimes x_i + \sum_{i=n+1}^m e_i \otimes x_i \right\|^2 \leq \left\| \sum_{i=1}^n e_i \otimes x_i \right\|^2 + \left\| \sum_{i=n+1}^m e_i \otimes x_i \right\|^2$$

for  $n, m \in \mathbb{N}$ , where  $(e_i)$  is the canonical orthonormal basis of  $\ell^2$ . It is easy to see that such a definition is equivalent to [70, Definition 2.1]. Let us denote by  $\mathcal{K}$  the algebra of compact operators on  $B(\ell^2)$ . A linear

map  $\phi : X \rightarrow Y$  between operator sequence spaces is *sequentially contractive* if  $id_K \otimes \phi$  is a contraction, and a *sequential isometry* if  $id_K \otimes \phi$  is an isometry. Operator sequence spaces have been introduced and studied in [69, 70]. They have been used in [70] to shed light on the properties of Figá-Talamanca–Hertz algebras. A systematic study of operator sequence spaces is presented in [69]. Every operator sequence space is canonically endowed with a *minimal operator space structure* [70, Definition 3.1]. We can regard operator sequence spaces as operator spaces endowed with their minimal operator space structure. It is proved in [69] that the operator spaces arising in this way are precisely the subspaces of  $\infty$ -sums of column operator Hilbert spaces [39, Subsection 3.4].

We can therefore regard the category  $\mathcal{A}$  of operator sequence spaces and sequential contractions as a full subcategory of the category of operator spaces and completely contractive maps. The collection  $\mathcal{I}$  of finite  $\infty$ -sum of finite-dimensional column operator Hilbert spaces is a collection of injective objects of  $\mathcal{A}$  that satisfies Conditions (1) and (2) of Subsection 2.3. Again the notion of basic tuple is provided by independent tuples. We can therefore conclude that finite-dimensional operator sequence spaces form a Fraïssé class. We call the corresponding limit  $\mathbb{CG}$  the *column Gurarij space*. The same argument as for  $p$ -multinormed spaces shows that the first order theory of  $\mathbb{CG}$  has a unique separable model and it admits elimination of quantifiers. As a consequence the Polish group of surjective complete isometries of  $\mathbb{CG}$  endowed with the topology of pointwise convergence is Roelcke precompact. As before,  $\mathbb{CG}$  is the unique existentially closed operator sequence space, and the theory of  $\mathbb{CG}$  is the model completion of the theory of operator sequence spaces. For a separable operator sequence space  $X$ , the following assertions are equivalent:

- $X$  is approximately injective in the sense of Definition 3.2;
- $X$  is sequentially isometric to the range of a sequentially contractive projection on  $\mathbb{CG}$ ;
- the identity map of  $X$  is the pointwise limit of multicontractive maps that factor through finite  $\infty$ -sums of finite-dimensional column operator Hilbert spaces;
- $X$  is a nuclear operator space with its canonical minimal operator space structure;
- $X$  is positively existentially closed in the class of operator sequence spaces.

**6.6.  $M_q$ -spaces.** Fix  $q \in \mathbb{N}$  and let  $M_q(\mathbb{C})$  be the space of complex  $q \times q$  matrices. If  $X$  is a complex vector space, then the algebraic tensor product  $M_q(\mathbb{C}) \otimes X$  can be canonically identified with the space  $M_q(X)$  of  $q \times q$  matrices with entries in  $X$ . There is a natural bimodule action of  $M_q(\mathbb{C})$  on  $M_q(\mathbb{C}) \otimes X$ . An  $M_q$ -space is a complex vector space such that  $M_q(\mathbb{C}) \otimes X$  is endowed with a norm satisfying

$$\left\| \sum_{i=1}^n \alpha_i^* x_i \beta_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i^* \alpha_i \right\| \max_{1 \leq i \leq n} \|x_i\| \left\| \sum_{i=1}^n \beta_i^* \beta_i \right\|$$

for  $\alpha_i \in M_q(\mathbb{C})$  and  $x_i \in M_q(\mathbb{C}) \otimes X$ , where the norms of complex  $q \times q$  matrices are the operator norms. Such spaces have been introduced and studied in [75], and subsequently used in [93, 91, 81]. For  $q = 1$  one obtains the class of complex Banach spaces. The language for  $M_q$ -spaces contains function symbols for the functions  $(x_1, \dots, x_n) \mapsto \alpha_1 x_1 \beta_1 + \dots + \alpha_n x_n \beta_n$  for every  $n \in \mathbb{N}$  and  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in M_q(\mathbb{Q}(i))$  such that  $\|\sum_{i=1}^n \alpha_i^* \alpha_i\| \leq 1$  and  $\|\sum_{i=1}^n \beta_i^* \beta_i\| \leq 1$ .

A linear map  $\phi : X \rightarrow Y$  between  $M_q$ -spaces is  $q$ -contractive if  $id_{M_q} \otimes \phi$  is contractive, and  $q$ -isometric if  $id_{M_q} \otimes \phi$  is isometric. The category of  $M_q$ -spaces also has products, given by  $\infty$ -sums [75, Subsection I.2.2]. Any  $M_q$ -space is  $q$ -isometric to a subspace of  $C(K, M_q(\mathbb{C})) = M_q(C(K))$  for some compact Hausdorff space  $K$  [75, Théorème I.1.9]. Proposition I.1.16 of [75] shows that  $M_q$  is an injective object in the category of  $M_q$ -spaces and  $q$ -contractions. Using these facts, one can easily show that the category  $\mathcal{A}$  of  $M_q$ -spaces and  $q$ -contractive maps, together with the collection  $\mathcal{I} \subset \mathcal{A}$  of finite  $\infty$ -sums of copies of  $M_q(\mathbb{C})$ , satisfy the assumptions of Section 2 with  $\varpi(\delta) = \delta$ . Hence all the results from Sections 2, 3, 4, and 5 apply in this setting. The corresponding limit  $\mathbb{G}_q$  has analogous property as  $\mathbb{G}$ ; see [49, §3].

**6.7.  $M_q$ -system.** An  $M_q$ -system is an  $M_q$ -space  $X$  with a distinguished element 1 (the *unit*) such that there exists a compact Hausdorff space  $K$  and a completely isometric linear map  $\phi : X \rightarrow C(K, M_q)$  which is *unital* in the sense that maps 1 to the function constantly equal to the identity matrix of  $M_q$ . Any  $M_q$ -system is endowed with an involution  $x \mapsto x^*$  coming from the inclusion into  $C(K, M_q)$ . Any unital linear contraction is automatically *self-adjoint*, that is commutes with the involution. These spaces have been introduced and studied in [110] under the name of  $q$ -minimal operator systems. For  $q = 1$  one obtains the notion of *complex function system* [97], which is the complex analog of the notion of (real) function systems as in Subsection 6.3.

As in the case of function systems,  $M_q$ -systems are in functorial 1:1 correspondence with natural geometric objects that we call  $M_q$ -convex sets. Suppose that  $V$  is a locally convex topological vector space  $V$ , and

$K_n \subset M_n(V)$  are compact convex sets for  $n = 1, 2, \dots, q$ . An  $M_q$ -convex combination is an expression of the form  $\alpha_1^* v_1 \alpha_1 + \dots + \alpha_\ell^* v_\ell \alpha_\ell$  where  $\alpha_i \in M_{n_i, q}$  and  $v_i \in K_{n_i}$  for  $1 \leq n_i \leq q$  and  $1 \leq i \leq \ell$ . We say that  $(K_1, \dots, K_q)$  is an  $M_q$ -convex set if it is closed under  $M_q$ -convex combinations. The notion of  $M_q$ -affine function between  $M_q$ -convex sets is defined in the obvious way by considering  $M_q$ -convex combinations rather than usual matrix convex combinations. To any  $M_q$ -convex set one can associate the  $M_q$ -system  $A(K_1, \dots, K_q)$  of real-valued  $M_q$ -convex affine functions, endowed with its canonical  $M_q$ -system structure. Conversely, any  $M_q$ -system  $X$  arises in this way from the  $M_q$ -convex set  $(S_1(X), \dots, S_q(X))$ , where  $S_n(X)$  is the space of  $q$ -contractive linear maps from  $X$  to  $M_n$ . Indeed such a correspondence is a particular case of the correspondence between operator systems and matrix convex sets established in [106].

We regard  $M_q$ -systems as structures in the language of  $M_q$ -spaces with the addition of a constant symbol for the unit and a unary function symbol for the involution. Again one can show that the category  $\mathcal{A}$  of  $M_q$ -systems and unital  $q$ -contractive maps, and the collection  $\mathcal{I} \subset \mathcal{A}$  of finite  $\infty$ -sums of copies of  $M_q$  satisfy the assumptions of Section 2. To see this one can use the small perturbation lemma [98, Lemma 2.13.2] together with the fact that approximately unital approximately  $q$ -isometric maps are close to unital  $q$ -contractive maps. This follows from the more general Lemma 8.4. In this context a basic tuple is a tuple  $\bar{a}$  of linearly independent elements such that the first element of the tuple  $\bar{a}$  is the unit. One then can infer that the conclusions of Section 2, 3, 4, and 5 hold in the setting of  $M_q$ -systems.

The limit of the class of finite-dimensional  $M_q$ -systems is an  $M_q$ -system  $A(\mathbb{P}_1^{(q)}, \dots, \mathbb{P}_q^{(q)})$ . It follows from the general results of this paper that—as shown in [49, Section 3]—the first order theory of  $A(\mathbb{P}_1^{(q)}, \dots, \mathbb{P}_q^{(q)})$  has a unique separable model and admits quantifier elimination. Applying Corollary 5.9 one can conclude that the action of  $\text{Aut}(A(\mathbb{P}_1^{(q)}, \dots, \mathbb{P}_q^{(q)}))$  on  $\mathbb{P}_1^{(q)}$  is minimal using the following lemma, which can be proved similarly as Lemma 8.10.

**Lemma 6.25.** *Suppose that  $q, d \in \mathbb{N}$  and  $\varepsilon > 0$ . There exists  $n \in \mathbb{N}$  such that for any states  $s$  on  $\ell_d^\infty(M_q)$  and  $t$  on  $\ell_n^\infty(M_q)$  there exists an embedding  $\phi : \ell_d^\infty(M_q) \rightarrow \ell_n^\infty(M_q)$  such that  $\|t \circ \phi - s\| < \varepsilon$ .*

When  $q = 1$  one recovers the Poulsen simplex  $\mathbb{P} = \mathbb{P}_1^{(1)}$ . The sequence of spaces  $(\mathbb{P}_1^{(q)}, \dots, \mathbb{P}_q^{(q)})$  for  $q \in \mathbb{N}$  can be seen as a sequence interpolating between the Poulsen simplex  $\mathbb{P}$  and the noncommutative Poulsen simplex  $\text{N}\mathbb{P}$ ; see 8.2.

## 7. MORE GENERAL FRAÏSSÉ CLASSES

**7.1. Stratified Fraïssé classes.** In this section we discuss how the framework of Section 2 can be generalized to apply to other classes of structures from functional analysis, such as the classes of exact operator spaces and exact operator systems. We still assume that  $\mathcal{L}$  is a countable language in the logic for metric structures, and keep the same notation and terminology as in Subsection 2.1.

Suppose that  $\mathcal{A}$  is a category of  $\mathcal{L}$ -structures with morphisms defined as in Definition 2.1. Let also  $\mathcal{A}_q \subset \mathcal{A}$  for  $q \in \mathbb{N} \cup \{\infty\}$  be a full subcategory such that  $\mathcal{A}_q \subset \mathcal{A}_{q+1}$ , and  $\mathcal{I}_q \subset \mathcal{A}_q$  be a countable collection of separable injective structures closed under finite products such that  $\mathcal{I}_q \subset \mathcal{I}_{q+1}$  and  $\mathcal{I}_\infty = \bigcup_q \mathcal{I}_q$ . Let  $\mathcal{I} \subset \mathcal{I}_\infty$  be a cofinal collection, that is such that any structure in  $\mathcal{I}_\infty$  admits an embedding into a structure of  $\mathcal{I}$ . We will assume that

- $\mathcal{A}$  satisfies Conditions (1)–(4) of Subsection 2.2 where the assumption (3) that  $\mathcal{A}$  has arbitrary products is replaced by the hypothesis that  $\mathcal{A}$  has finite products;
- $\mathcal{A}_q$  satisfies all the conditions of Subsection 2.2;
- $\mathcal{A}_q$  has enough injectives from  $\mathcal{I}_q$  with modulus  $\varpi$  as in Definition 2.6;
- $\mathcal{A}$  has enough injectives from  $\mathcal{I}_\infty$  with modulus  $\varpi$  as in Definition 2.6.

The arguments of Section 2 apply in this more general situation to show the following.

**Theorem 7.1.** *The class of finitely generated structures in  $\mathcal{A}$  is a Fraïssé class. The corresponding limit  $M$  can be realized as limit of a direct sequence of structures in  $\mathcal{I}$  with embeddings as connective maps. Furthermore the limit admits the same characterization as in Proposition 2.12.*

Also the arguments of Section 4 and Section 5 go through in this more general setting. This yields a generic morphism  $\Omega_M^R : M \rightarrow R$  for any approximately injective separable structure  $R$  in  $\mathcal{A}$ , and a generic operator  $\Omega_M : M \rightarrow M$ , with the same characterization and properties as in Section 4.



**7.2. Approximately injective objects and retracts of the limit.** The class of approximately injective structures in  $\mathcal{A}$  can be defined similarly as in Definition 3.2. Again, since the elements of  $\mathcal{I}$  are injective, and the Fraïssé limit  $M$  of the class of finitely generated structures in  $\mathcal{A}$  is the limit of an inductive sequence of elements of  $\mathcal{I}$  with embeddings as connective maps, it follows that  $M$  is approximately injective. As a consequence the retracts of  $M$  are approximately injective as well. The following theorem shows that, conversely, any separable approximately injective structure in  $\mathcal{A}$  is isomorphic to a retract of  $M$ .

**Theorem 7.2.** *Let  $M$  denote the Fraïssé limit of the class of finitely generated structures in  $\mathcal{A}$ . A separable structure  $X$  in  $\mathcal{A}$  is approximately injective if and only if there exist an embedding  $\eta : X \rightarrow M$  and an idempotent morphism  $\pi : M \rightarrow M$  such that the range of  $\eta$  equals the range of  $\pi$ .*

*Proof.* Suppose that  $X$  is a separable approximately injective structure in  $\mathcal{A}$ . Our aim is to construct a separable structure  $Z$  in  $\mathcal{A}$ , an embedding  $\eta : X \rightarrow Z$ , and a morphism  $\pi : Z \rightarrow X$  such that  $\pi \circ \eta$  is the identity map of  $X$  and  $Z$  is the Fraïssé limit of the class of finitely generated elements of  $\mathcal{A}$ .

Recall that if  $f, g : A \rightarrow B$  are morphisms between structures in  $\mathcal{A}$ ,  $\bar{a}$  is a tuple in  $A$ , and  $\varepsilon > 0$ , then we write  $f \approx_{\bar{a}, \varepsilon} g$  to indicate that  $d(f(\bar{a}), g(\bar{a})) < \varepsilon$ . Fix an enumeration  $\{A_{q,m} : m \in \mathbb{N}\}$  of the elements of  $\mathcal{I}_q$ . Fix for every  $m \in \mathbb{N}$  a countable fundamental subset  $D_{q,m}$  of  $A_{q,m}$  as in Definition 2.5 and an enumeration  $\bar{a}_{q,m,i}$  of the basic tuples in  $D_{q,m}$ . Let  $E_{q,m,i} = \langle \bar{a}_{q,m,i} \rangle$ . One can build by recursion on  $n$

- an increasing sequence  $(q_n)$  in  $\mathbb{N}$  such that  $q_n \geq n$ ,
- an increasing sequence  $(X_n)$  of substructures of  $X$  with dense union,
- basic generating tuples  $\bar{b}_n$  of  $X_n$ ,
- a sequence  $(\varepsilon_n)$  of strictly positive real numbers such that  $\varpi(\varepsilon_{n+1}) \leq \varepsilon_n \leq 2^{-2n}$ ,
- a sequence  $(\hat{X}_n)$  of finitely generated structures in  $\mathcal{A}$  such that  $\hat{X}_n \in \mathcal{A}_{q_n}$ ,
- morphisms  $\varphi_n : X_n \rightarrow \hat{X}_n$ ,  $\psi_n : \hat{X}_n \rightarrow X_n$ , and  $\hat{i}_n : \hat{X}_n \rightarrow \hat{X}_{n+1}$  such that, if  $i_n : X_n \rightarrow X_{n+1}$  denotes the inclusion map,  $I(\varphi_n) < \varepsilon_n$ ,  $I(\psi_n) < \varepsilon_n$ ,  $d(\psi_n \circ \varphi_n, id_{X_n}) < \varepsilon_n$ ,  $\hat{i}_{n+1} = \varphi_{n+1} \circ i_n \circ \psi_n$ , and hence  $d(\hat{i}_{n+1} \circ \varphi_n, \varphi_{n+1} \circ i_n) < \varepsilon_n$ ,
- a direct sequence  $(Z_n)$  of finitely generated structures in  $\mathcal{A}$  with embeddings  $j_n : Z_n \rightarrow Z_{n+1}$  as connective maps,
- embeddings  $\eta_n : \hat{X}_n \rightarrow Z_n$  such that  $\eta_{n+1} \circ \hat{i}_n \circ \varphi_n \approx_{\bar{b}_n, \varepsilon_n} j_n \circ \eta_n \circ \varphi_n$ ,
- morphisms  $\pi_n : Z_n \rightarrow \hat{X}_n$  such that  $\pi_{n+1} \circ j_n = \hat{i}_n \circ \pi_n$  and  $\pi_{n+1} \circ \eta_{n+1} = id_{\hat{X}_{n+1}}$ ,
- a finite  $\varepsilon_n$ -dense set  $\mathcal{G}_n$  of morphisms  $g : E_{q,m,i} \rightarrow Z_n$  for  $m, i \leq n$  and  $q \leq q_n$  such that  $j_n \circ g \in \mathcal{G}_n$  for every  $g \in \mathcal{G}_{n-1}$ ,
- a finite  $\varepsilon_n$ -dense set  $\mathcal{F}_n$  of morphisms  $f : E_{q,m,i} \rightarrow X_n$  for  $m, i \leq n$  and  $q \leq q_n$  such that  $i_n \circ f \in \mathcal{F}_n$  for every  $f \in \mathcal{F}_{n-1}$  and  $\psi_n \circ \pi_n \circ g \in \mathcal{F}_n$  for every  $g \in \mathcal{G}_n$ ,

such that

- (1) for any  $i, m \leq n$ ,  $q \leq q_n$ , and morphism  $f : E_{q,m,i} \rightarrow X_n$  in  $\mathcal{F}_n$  there exists a morphism  $\hat{f} : A_{q,m} \rightarrow X_{n+1}$  such that  $\hat{f} \approx_{\bar{a}_{q,m,i}, \varepsilon_n} f$ ;
- (2) for any  $m, i \leq n$ ,  $q \leq q_n$ , and morphism  $g : E_{q,m,i} \rightarrow Z_n$  in  $\mathcal{G}_n$ , there exists an embedding  $\hat{g} : A_{q,m} \rightarrow Z_{n+1}$  such that  $d(\hat{g}(\bar{a}_{q,m,i}), g(\bar{a}_{q,m,i})) \leq \varpi(I(g) + \varepsilon_n)$ .

The construction proceeds as follows. Fix an enumeration  $\{w_n : n \in \mathbb{N}\}$  of a dense subset of  $X$ . Set  $X_1 = \langle w_1 \rangle$ . Using Condition (1) of Subsection 2.3 for the classes  $\mathcal{A}$  and  $\mathcal{I}_\infty$  one can find  $q_1 \in \mathbb{N}$ , a finitely generated structure  $\hat{X}_1$  in  $\mathcal{A}_{q_1}$ , and morphisms  $\varphi_1 : X_1 \rightarrow \hat{X}_1$  and  $\psi_1 : \hat{X}_1 \rightarrow X_1$  such that  $I(\varphi_1) < \varepsilon_1$ ,  $I(\psi_1) < \varepsilon_1$ ,  $I(\varphi_1) < \varepsilon_1$ ,  $d(\psi_1 \circ \varphi_1, id_{X_1}) < \varepsilon_1$ , and  $d(\varphi_1 \circ \psi_1, id_{\hat{X}_1}) < \varepsilon_1$ .

Suppose now that, by induction hypothesis,  $X_k, \bar{b}_k, q_k, \hat{X}_k, \varphi_k, Z_k, \eta_k, \pi_k, \varepsilon_k, \mathcal{F}_k, \mathcal{G}_k, j_{k-1}$  have been defined for  $k \leq n$ . Since Condition (1) only involves finitely many morphisms  $f$ , one can find  $\bar{b}_{n+1} \supset \bar{b}_n \cup \{w_{n+1}\}$  such that  $X_{n+1} = \langle \bar{b}_{n+1} \rangle$  satisfies (1) by repeatedly applying the assumption that  $X$  is approximately injective. One can then find  $\hat{X}_{n+1}, \varphi_n, \psi_n, q_{n+1}$  reasoning as for  $\hat{X}_1, \varphi_1, \psi_1$ . Let now  $Z_{n+1}$  be the approximate pushout within the class  $\mathcal{A}_{q_{n+1}}$  constructed as in Lemma 3.1 of the maps  $\eta_n \circ \varphi_n : X_n \rightarrow Z_n$  and  $\varphi_{n+1} \circ i_n : X_n \rightarrow \hat{X}_{n+1}$  over  $\bar{b}_n$  with tolerance  $\varepsilon_n$ , and of the maps  $f : E_{q,m,i} \rightarrow Z_n$  and  $E_{q,m,i} \hookrightarrow A_{q,m}$  (inclusion map) over  $\bar{a}_{q,m,i}$  with tolerance  $\varpi(I(f) + \varepsilon_n)$ , where  $m, i \leq n$ ,  $q \leq q_n$  are such that  $E_{q,m,i} \subset A_{q,m}$  and  $f : E_{q,m,i} \rightarrow Z_n$  is a morphism in  $\mathcal{F}_n$ . Let also  $j_n : Z_n \rightarrow Z_{n+1}$  and  $\eta_{n+1} : \hat{X}_{n+1} \rightarrow Z_{n+1}$  be the canonical morphisms of the approximate pushout, and observe that by definition  $\eta_{n+1} \circ \varphi_{n+1} \circ i_n \approx_{\bar{b}_n, \varepsilon_n} j_n \circ \eta_n \circ \varphi_n$ . We want to define a morphism  $\pi_{n+1} : Z_{n+1} \rightarrow \hat{X}_{n+1}$ . By inductive hypothesis we have that  $\hat{i}_n \circ \pi_n : Z_n \rightarrow \hat{X}_{n+1}$  and  $\varphi_{n+1} : X_{n+1} \rightarrow \hat{X}_{n+1}$  are morphisms such that

$$\hat{i}_n \circ \pi_n \circ \eta_n \circ \varphi_n = \hat{i}_n \circ \varphi_n \approx_{\bar{b}_n, \varepsilon_n} \varphi_{n+1} \circ i_n.$$

Furthermore if  $m, i \leq n$  and  $q \leq q_{n+1}$  are such that  $E_{q,m,i} \subset A_{q,m}$  and  $g : E_{q,m,i} \rightarrow Z_n$  is a morphism in  $\mathcal{G}_n$ , then  $f := \psi_n \circ \pi_n \circ g : E_{q,m,i} \rightarrow X_n$  is a morphism in  $\mathcal{F}_n$ . Therefore by inductive hypothesis there exists a morphism  $\hat{f} : A_{q,m} \rightarrow X_{n+1}$  such that  $\hat{f} \approx_{\bar{a}_{q,m,i}, \varepsilon_n} f$ . Hence  $\varphi_{n+1} \circ \hat{f} : A_{q,m} \rightarrow \hat{X}_{n+1}$  is a morphism such that

$$\varphi_{n+1} \circ \hat{f} \approx_{\bar{a}_{q,m,i}, \varepsilon_n} \varphi_{n+1} \circ f = \varphi_{n+1} \circ \psi_n \circ \pi_n \circ g = \hat{i}_n \circ \pi_n \circ g.$$

Therefore by the universal property of the approximate pushout there exists a morphism  $\pi_{n+1} : Z_{n+1} \rightarrow \hat{X}_{n+1}$  such that  $\pi_{n+1} \circ j_n = \hat{i}_n \circ \pi_n$  and  $\pi_{n+1} \circ \eta_{n+1} = id_{\hat{X}_{n+1}}$ . This concludes the recursive construction. Granted the construction one can then define  $Z$  to be the limit in  $\mathcal{A}$  of the inductive sequence  $(Z_n)$  with connecting maps  $j_n$ . Let  $\eta$  be the embedding of  $X$  into  $Z$  obtained as the limit of the sequence  $\eta_n \circ \varphi_n : X_n \rightarrow Z_n$ . Finally let  $\pi : Z \rightarrow X$  be the morphism obtained as the limit of the sequence  $\psi_n \circ \pi_n : Z_n \rightarrow X_n$ . It follows from the properties of the maps  $\eta_n, \varphi_n, \psi_n, \pi_n$  listed above that  $\eta$  and  $\pi$  are well defined and satisfy  $\pi \circ \eta = id_X$ . Furthermore the assumption (2) in the construction guarantees that  $Z$  is the Fraïssé limit of the class of finite dimensional structures in  $\mathcal{A}$ .  $\square$

Using Theorem 7.2 one can also prove that the approximately injective structures in  $\mathcal{A}$  are precisely the  $\mathcal{I}$ -nuclear structures as in Definition 3.4; see the proof of Proposition 3.5.

One can alternatively prove Theorem 7.2 using the construction from Subsection 5.3 generalized to the setting of stratified Fraïssé classes generated by injective objects. Indeed if  $X$  is a separable approximately injective structure in  $\mathcal{A}$ , there exist a morphism  $\Omega_M^X : M \rightarrow X$  and an embedding  $\eta_M^X : X \rightarrow M$  such that  $\Omega_M^X \circ \eta_M^X$  is the identity of  $X$ . Thus  $\eta_M^X \circ \Omega_M^X : X \rightarrow X$  is a retraction of  $X$  onto a substructure isomorphic to  $M$ .

## 8. MORE EXAMPLES

**8.1. Exact operator spaces.** Let  $\mathcal{K}$  be the space of compact linear operators on  $\ell^2$ . If  $X$  is a complex vector space, then the space  $\mathcal{K} \otimes X$  is naturally endowed with a  $\mathcal{K}$ -bimodule structure. An *operator space* is a complex vector space  $X$  such that  $\mathcal{K} \otimes X$  is endowed with a norm satisfying

$$\left\| \sum_{i=1}^n \alpha_i^* x_i \beta_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i^* \alpha_i \right\| \max_{1 \leq i \leq n} \|x_i\| \left\| \sum_{i=1}^n \beta_i^* \beta_i \right\|$$

where  $n \in \mathbb{N}$ ,  $\alpha_i, \beta_i \in \mathcal{K}$ , and  $x_i \in \mathcal{K} \otimes X$ . A linear map  $\phi : X \rightarrow Y$  between operator spaces is *completely contractive* if  $id_{\mathcal{K}} \otimes \phi$  is contractive, and completely isometric if  $id_{\mathcal{K}} \otimes \phi$  is isometric.

As before, we let  $\mathcal{K}_0(\mathbb{Q}(i))$  be the space of finite rank operators whose coefficients with respect to the canonical basis of  $\ell^2$  belong to the field of Gauss rationals  $\mathbb{Q}(i)$ . Let  $\mathcal{L}$  be the language containing an  $n$ -ary function symbol  $f_{\bar{\alpha}, \bar{\beta}}$  for every  $n \in \mathbb{N}$  and  $n$ -tuples  $\bar{\alpha}$  and  $\bar{\beta}$  in  $\mathcal{K}_0(\mathbb{Q}(i))$  such that  $\|\sum_{i=1}^n \alpha_i^* \alpha_i\| \leq 1$  and  $\|\sum_{i=1}^n \beta_i^* \beta_i\| \leq 1$ . If  $X$  is an operator space, then one can regard  $X$  as an  $\mathcal{L}$ -structure with support the unit ball of  $\mathcal{K} \otimes X$ , where the interpretation of  $f_{\bar{\alpha}, \bar{\beta}}$  is the function

$$(x_1, \dots, x_n) \mapsto \alpha_1 x_1 \beta_1 + \dots + \alpha_n x_n \beta_n.$$

It is clear that under this identification a morphism in the sense of Subsection 2.1 is (the restriction to the unit ball of) a completely contractive linear map, and an embedding is (the restriction to the unit ball of) a completely isometric linear map. It is not hard to verify that if  $f : X \rightarrow Y$  is a completely contractive linear map between operator spaces and  $0 \leq \delta \leq 1$ , then  $I(f) \leq \delta$  if and only if  $\|id_{\mathcal{K}} \otimes f^{-1}\| \leq 1 + \delta$ .

Suppose that  $H$  is a Hilbert space. Denote by  $B(H)$  the algebra of bounded linear operators on  $H$  endowed with the operator norm. If  $X$  is a linear subspace of  $B(H)$ , then  $X$  has a natural operator space structure obtained by identifying  $M_n(X)$  with a subspace of the algebra  $B(H^{\oplus n})$  of bounded linear operators on the  $n$ -fold Hilbertian direct sum of  $H$  by itself. Conversely an operator space is linearly completely isometric to a space of this form [102]. We denote by  $M_{d,k}(\mathbb{C})$  the operator space of  $d \times k$  matrices, identified with the space  $B(H, K)$  of bounded linear operators from a  $k$ -dimensional Hilbert space  $H$  to a  $d$ -dimensional Hilbert space  $K$ . By the Arveson-Wittstock-Paulsen extension theorem [96, Theorem 8.2] and the main result of [104], the finite-dimensional *injective* operator spaces are precisely the finite  $\infty$ -sums of copies of  $M_{d,k}(\mathbb{C})$  for  $d, k \in \mathbb{N}$ . These are also precisely the finite-dimensional *ternary rings of operators*; see [59]. When  $k = d$  we simply write  $M_d(\mathbb{C})$ .

An operator space  $X$  is called *exact* if for any  $\delta > 0$  and for any finite-dimensional subspace  $E$  of  $X$  there exists  $n \in \mathbb{N}$  and a completely contractive linear map  $f : X \rightarrow M_n(\mathbb{C})$  such that  $\|id_{\mathcal{K}} \otimes f^{-1}\| \leq 1 + \delta$ . If  $X$  is

an  $M_q$ -space as in Subsection 6.6 then one can canonically endow  $X$  with an (exact) operator space structure  $\text{MIN}_q(X)$  defined by setting

$$\|x\| = \sup_{\phi} \|(id_{\mathcal{K}} \otimes \phi)(x)\|$$

for  $x \in \mathcal{K} \otimes X$ , where  $\phi$  ranges among all the  $q$ -contractions from  $X$  to  $M_q(\mathbb{C})$ .

Let now  $\mathcal{A}$  be the class of operator spaces,  $\mathcal{A}_q$  be the class of operator spaces of the form  $\text{MIN}_q(X)$  for some  $M_q$ -space  $X$ ,  $\mathcal{I}_q$  be the class of finite  $\infty$ -sums of copies of  $M_{d,k}(\mathbb{C})$  for  $d, k \leq q$  (these are precisely the finite-dimensional  $q$ -minimal injective operator spaces),  $\mathcal{I}_{\infty}$  be the union of  $\mathcal{I}_q$  for  $q \in \mathbb{N}$ , and  $\mathcal{I} \subset \mathcal{I}_{\infty}$  be the class of operator spaces of the form  $M_n(\mathbb{C})$  for  $n \in \mathbb{N}$ . The small perturbation lemma shows that, by declaring a tuple in an operator space basic if it is linearly independent, one obtains a notion of basic tuples that satisfies the assumptions of Subsection 2.2. The definition of exact operator spaces implies that the classes  $\mathcal{A}$  and  $\mathcal{I}$  satisfy Condition (1) of Subsection 2.3. Condition (2) of Subsection 2.3 with  $\varpi(\delta) = \delta$  is easily verified by considering the composition of  $f$  with the inverse map of  $\phi$  (when  $\phi$  is injective) and then normalizing. The operator spaces that are approximately injective according to Definition 3.2 are precisely the nuclear operator spaces; see [39, §14.6]. Similarly the operator spaces that are rigid  $\mathcal{I}_{\infty}$ -structures as in Definition 3.6 are precisely the rigid rectangular  $\mathcal{OL}_{\infty,1+}$  spaces [56, §2]. It follows from [81, Proposition 5.15] that the separable rigid rectangular  $\mathcal{OL}_{\infty,1+}$  spaces are precisely the operator spaces that can be written as limits of inductive sequences of finite-dimensional injective operator spaces with completely isometric connective maps. Not every nuclear operator space is rigid rectangular  $\mathcal{OL}_{\infty,1+}$  space. An example is the Cuntz  $C^*$ -algebra  $\mathcal{O}_2$  [29, §V.4].

One can then apply the conclusions of Section 7 to prove that the class of finite-dimensional exact operator spaces form a Fraïssé class, recovering a result from [81]. The corresponding limit is the Gurarij operator space  $\text{NG}$  introduced in [91] and proved to be unique in [81]. Theorem 3.3 implies that a separable exact operator space is nuclear if and only if it is completely isometric to the range of a completely contractive projection of  $\text{NG}$ . This recover a result from [80]. The existence of the universal operator on  $\text{NG}$  described in Theorem 4.3 follows from considering the class of completely contractive linear maps between finite-dimensional exact operator spaces, as discussed in Subsection 4.3 and Subsection 7.2. The model-theoretic properties of  $\text{NG}$  have been considered in [81, Section 5.8], building on [49], where it is shown among other things that  $\text{NG}$  is the unique separable exact existentially closed operator space and the prime model of its first order theory. An operator space is nuclear if and only if it is positively existentially closed.

The noncommutative analogs of  $M$ -ideals in Banach spaces are the complete  $M$ -ideals in operator spaces introduced in [38]. It is proved in [38, Proposition 4.4] that a subspace  $N$  of an operator space  $Z$  is a complete  $M$ -ideal if and only if  $M_n(N)$  is an  $M$ -ideal of  $M_n(Z)$  for every  $n \in \mathbb{N}$ . The following is the natural noncommutative analog of the notion of facial quotient from Definition 6.7.

**Definition 8.1.** *A complete facial quotient mapping  $P : Z \rightarrow X$  between operator spaces is a complete quotient mapping whose kernel is a complete  $M$ -ideal.*

It is clear that when  $Z, X$  are Banach spaces endowed with the canonical minimal operator space structure, then  $P : Z \rightarrow X$  is a complete facial quotient if and only if it is a facial quotient.

If  $\mathbf{K}$  is a compact rectangular matrix convex set as in the introduction—see also [44, Section 3]—then one can define the notion of closed rectangular matrix face of  $\mathbf{K}$  in terms of complete facial quotients. By definition, a closed rectangular convex subset  $\mathbf{F}$  of  $\mathbf{K}$  is a closed rectangular matrix face whenever the associated restriction mapping  $A_{\sigma}(\mathbf{K}) \rightarrow A_{\sigma}(\mathbf{F})$  is a complete facial quotient.

Recall that an operator space  $X$  satisfies the operator metric approximation property if the identity map of  $X$  is the pointwise limit of finite rank completely contractive linear maps [37]. The following characterization of complete facial quotients is the natural noncommutative analog of Proposition 6.21.

**Proposition 8.2.** *Suppose that  $X, Y$  are operator spaces, and  $P : Z \rightarrow X$  is a complete quotient map. The following statements are equivalent:*

- (1)  *$P$  is a complete facial quotient;*
- (2) *whenever  $\varepsilon > 0$ ,  $E \subset F$  are finite-dimensional operator spaces,  $g : F \rightarrow X$  is a linear complete contraction, and  $f : E \rightarrow Z$  is a linear complete isometry such that  $P \circ f = g|_E$ , then there exists a linear complete contraction  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} \leq \varepsilon$ ;*
- (3) *whenever  $\varepsilon > 0$ ,  $A$  is a separable operator space with the operator metric approximation property,  $E \subset A$  is a finite-dimensional subspace, and  $f : E \rightarrow Z$  and  $g : A \rightarrow X$  are linear complete contractions such that  $\|P \circ f - g|_E\|_{cb} < \varepsilon$ , then there exists a linear complete contraction  $\hat{g} : A \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} < 6\varepsilon$ ;*

*If furthermore  $Z$  is exact and  $X$  is nuclear, then these are also equivalent to:*

- (4) for any  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , finite-dimensional  $q$ -minimal operator spaces  $E \subset F$ , linear complete contractions  $f : E \rightarrow Z$  and  $g : F \rightarrow X$  such that  $P \circ f = g|_E$ , there exists a linear complete contraction  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} \leq \varepsilon$ .

*Proof.* The implication (1) $\Rightarrow$ (3) can be proved as [38, Theorem 5.2]. The implications (3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) We denote by  $N$  the kernel of  $P$ . Fix  $n \in \mathbb{N}$ . It is enough to prove that  $M_n(N)$  is an  $M$ -ideal of  $M_n(Z)$ . Fix  $\varepsilon > 0$ ,  $y^{(1)} = [y_{ij}^{(1)}], y^{(2)} = [y_{ij}^{(2)}], y^{(3)} = [y_{ij}^{(3)}] \in M_n(N)$  and  $x = [x_{ij}] \in M_n(Z)$  such that  $\max\{\|y^{(1)}\|, \|y^{(2)}\|, \|y^{(3)}\|, \|x\|\} \leq 1$ . In view of the implication (iv) $\Rightarrow$ (i) in [53, Theorem 2.2], it is enough to prove that there exists  $y \in M_n(N)$  such that  $\|y\| \leq 1$  and  $\|x + y^{(\ell)} - y\| \leq 1 + \varepsilon$  for  $\ell \in \{1, 2, 3\}$ . Consider

$$E = \text{span}\{y_{ij}^{(k)}, x_{ij} : i, j \leq n\} \subset Z.$$

We denote by  $e_{ij}$  the matrix units of  $M_n(\mathbb{C})$  and by  $e$  the element  $[e_{ij}]$  of  $M_n(M_n(\mathbb{C}))$ . Let  $F$  be the operator space obtained from  $E \oplus M_n(\mathbb{C})$  and the collection of linear maps  $E \oplus M_n(\mathbb{C}) \rightarrow B(H)$  of the form  $(z, \alpha) \mapsto \varphi(z) + \psi(\alpha)$  where  $\varphi : E \rightarrow B(H)$  is completely contractive,  $\psi : M_n(\mathbb{C}) \rightarrow B(H)$  is such that  $\|\psi^{(n)}(e)\| \leq 1$ , and  $\|\varphi^{(n)}(x + y^{(\ell)}) - \psi^{(n)}(e)\| \leq 1$ . By definition we have that the norm of  $(x + y^{(\ell)}, -e)$  evaluated in  $M_n(F)$  is at most 1. We observe that the canonical inclusion  $E \subset F$  is completely isometric. Indeed if  $k \in \mathbb{N}$  and  $z \in M_k(E)$  is such that  $\|z\| = 1$  then there exists a completely contractive map  $\varphi : E \rightarrow B(H)$  such that  $\|\varphi^{(k)}(z)\| = 1$ . Define  $\psi : M_n(\mathbb{C}) \rightarrow B(H)$ ,  $e_{ij} \mapsto \varphi(x_{ij})$ . The maps  $\varphi$  and  $\psi$  witness that the image of  $z$  inside  $M_k(F)$  has norm 1. This concludes the proof that the inclusion  $E \subset F$  is completely isometric. Define now the map  $g : F \rightarrow X$  by mapping  $(z, \alpha)$  to  $P(z)$ . Observe that  $g$  is completely contractive. Indeed if  $k \in \mathbb{N}$ ,  $z \in M_k(X)$ , and  $\alpha \in M_k(\mathbb{C})$ , pick a completely contractive map  $\rho : X \rightarrow B(H)$  such that  $\|(\rho \circ P)(z)\| = \|P(z)\|$ . Then the maps  $\varphi := (\rho \circ P)|_E$  and  $\psi = 0$  witness that  $\|P(z)\|$  is smaller than or equal to the norm of  $(z, \alpha)$  evaluated in  $F$ . This shows that the map  $g$  is completely contractive. Applying our assumption to the map  $g$  and the inclusion map  $f : E \rightarrow Z$  one obtains a completely contractive map  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} \leq \varepsilon$ . Set now  $y_{ij} = \hat{g}(e_{ij})$  for  $i, j \leq n$  and  $y = [y_{ij}] \in M_n(Z)$ . We have that for  $\ell \in \{1, 2, 3\}$ ,

$$\begin{aligned} \|x + y^{(\ell)} - y\|_{M_n(Z)} &= \|x + y^{(\ell)} - \hat{g}^{(n)}(e)\|_{M_n(Z)} \\ &\leq \|\hat{g}^{(n)}(x + y^{(\ell)} - e)\|_{M_n(Z)} + \varepsilon \leq \|(x + y^{(\ell)}, -e)\|_{M_n(F)} + \varepsilon \leq 1 + \varepsilon. \end{aligned}$$

This concludes the proof.

Suppose now that  $X, Z$  are rigid rectangular  $\mathcal{OL}_{\infty,1+}$  spaces.

(4) $\Rightarrow$ (1) As in the proof of (2) $\Rightarrow$ (1), we fix  $\varepsilon \in (0, 1]$ ,  $y^{(1)} = [y_{ij}^{(1)}], y^{(2)} = [y_{ij}^{(2)}], y^{(3)} = [y_{ij}^{(3)}] \in M_n(N)$  and  $x = [x_{ij}] \in M_n(Z)$  such that  $\max\{\|y^{(1)}\|, \|y^{(2)}\|, \|y^{(3)}\|, \|x\|\} \leq 1$ . We want to prove that there exists  $y \in M_n(N)$  such that  $\|y\| \leq 1$  and  $\|x + y^{(\ell)} - y\| \leq 1 + \varepsilon$  for  $\ell \in \{1, 2, 3\}$ . Define  $E \subset Z$  and  $e \in M_n(E)$  as in the proof of (2) $\Rightarrow$ (1). Fix  $\delta \in (0, \varepsilon/4]$ . For  $q \in \mathbb{N}$ , we denote by  $\text{MIN}_q(E)$  the space  $E$  endowed with its canonical  $q$ -minimal operator space structure; see [93, Section 2]. Define  $B$  to be the image of  $E$  under  $P$ , and  $\iota_B : B \rightarrow X$  the inclusion map. Since  $Z$  is exact and  $X$  is nuclear, there exist  $q \geq n$  and completely contractive maps  $\gamma : B \rightarrow M_q(\mathbb{C})$  and  $\rho : M_q(\mathbb{C}) \rightarrow Z$  such that  $\|\rho \circ \gamma - \iota_B\|_{cb} \leq \delta/2$  and the inclusion map  $\iota_E : \text{MIN}_q(E) \rightarrow Z$  has completely bounded norm at most  $1 + \delta$ . Let  $F$  be the  $q$ -minimal operator space obtained from  $E \oplus M_n(\mathbb{C})$  and the collection of linear maps  $E \oplus M_n(\mathbb{C}) \rightarrow M_q(\mathbb{C})$ ,  $(z, \alpha) \mapsto \varphi(z) + \psi(\alpha)$  such that  $\varphi : E \rightarrow M_q(\mathbb{C})$  is completely contractive,  $\|\psi^{(n)}(e)\| \leq 1$ , and  $\|\varphi^{(n)}(x + y^{(\ell)}) - \psi^{(n)}(e)\| \leq 1 + \varepsilon$ . Define also  $g : F \rightarrow X$  by  $g(z, \alpha) = \frac{1}{1+\delta}P(z)$  and  $f : \text{MIN}_q(E) \rightarrow Z$  by  $f := \frac{1}{1+\delta}\iota_E$ . Observe that the inclusion  $\text{MIN}_q(E) \subset F$  is completely isometric, the maps  $g : F \rightarrow X$  and  $f : \text{MIN}_q(E) \rightarrow Z$  are completely contractive, and  $P \circ f = g|_{\text{MIN}_q(E)}$ . Therefore by assumption there exists a completely contractive map  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_{\text{MIN}_q(E)} - f\|_{cb} \leq \delta$ . Set  $y := \hat{g}^{(n)}(e)$ . Hence we have for  $\ell \in \{1, 2, 3\}$ ,

$$\begin{aligned} \|x + y^{(\ell)} - y\|_{M_n(Z)} &\leq \left\| \frac{1}{1+\delta}(x + y^{(\ell)}) - \hat{g}^{(n)}(e) \right\|_{M_n(Z)} + 2\delta = \|f^{(n)}(x + y^{(\ell)}) - \hat{g}^{(n)}(e)\|_{M_n(Z)} + 2\delta \\ &\leq \|\hat{g}(x + y^{(\ell)} - e)\|_{M_n(Z)} + 4\delta \leq \|(x + y^{(\ell)}, -e)\|_{M_n(F)} + 4\delta \leq 1 + \varepsilon. \end{aligned}$$

This concludes the proof.  $\square$

Suppose that  $H, K$  are Hilbert spaces, and  $X \subset B(H, K)$  and  $Z$  are operator spaces. A *rectangular operator convex combination* as defined in [44] is an expression  $\alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  where  $\phi_i : Z \rightarrow B(H_i, K_i)$  are

completely contractive maps for some Hilbert spaces  $H_i, K_i$ , and  $\beta_i : H \rightarrow H_i$  and  $\alpha_i : K \rightarrow K_i$  are linear maps of norm at most 1. We say that  $\alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is a *proper* rectangular operator convex combination if  $\alpha_i, \beta_i$  are surjective,  $\alpha_1^* \alpha_1 + \cdots + \alpha_n^* \alpha_n = 1$ , and  $\beta_1^* \beta_1 + \cdots + \beta_n^* \beta_n = 1$ . A proper rectangular operator convex combination  $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is *trivial* if  $\alpha_i^* \alpha_i = \lambda_i 1$ ,  $\beta_i^* \beta_i = \lambda_i$ , and  $\alpha_i^* \phi_i \beta_i = \lambda_i \phi$  for some  $\lambda_i \in [0, 1]$ . A completely contractive map  $\phi : Z \rightarrow X$  such that  $\|\phi\|_{cb} = 1$  is a *rectangular operator extreme point* if any proper rectangular operator convex combination  $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is trivial. We observe that, if  $V$  is a finite-dimensional injective operator space, then the identity map  $V \rightarrow V$  is a rectangular operator extreme point. Indeed in this case  $V$  is a ternary ring of operators. The conclusion follows by passing to the linking algebra [59] and then applying [4, Corollary 1.4.3].

**Proposition 8.3.** *Suppose that  $Z$  and  $X$  are rigid rectangular  $\mathcal{OL}_{\infty,1+}$  spaces and  $\phi : Z \rightarrow X$  is a complete facial quotient. Then  $\phi$  is a rectangular operator extreme point.*

*Proof.* Consider  $X \subset B(H, K)$ . Suppose that  $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is a proper rectangular matrix convex combination as above. Fix  $\varepsilon > 0$  and a finite-dimensional injective operator space  $V \subset Z$ . Since  $X$  is a rigid  $\mathcal{OL}_{\infty,1+}$  space, we can find a finite-dimensional injective operator space  $W \subset X$  and a completely contractive map  $\psi : V \rightarrow W$  such that  $\|\psi - \phi\|_{cb} < \varepsilon$ . Consider now the complete isometry  $\eta : V \rightarrow V \oplus^\infty W$ ,  $x \mapsto (x, \psi(x))$  and the completely contractive map  $g : V \oplus^\infty W \rightarrow X$ ,  $(z, y) \mapsto y$ . Observe that  $g \circ \eta = \psi$ . Since  $\phi$  is a complete facial quotient, there exists a completely contractive map  $\hat{g} : V \oplus^\infty W \rightarrow Z$  such that  $\phi \circ \hat{g} = g$  and  $\|\hat{g} \circ \eta - \iota\|_{cb} < 6\varepsilon$ , where  $\iota : V \rightarrow Z$  is the inclusion map. We have that

$$g = \phi \circ \hat{g} = \alpha_1^* (\phi_1 \circ \hat{g}) \beta_1 + \cdots + \alpha_n^* (\phi_n \circ \hat{g}) \beta_n.$$

Since  $g$  is a rectangular operator extreme point, we can conclude that there exist  $\lambda_1, \dots, \lambda_n \in [0, 1]$  such that  $\alpha_i^* \alpha_i = \lambda_i 1$ ,  $\beta_i^* \beta_i = \lambda_i$ , and  $\alpha_i^* (\phi_i \circ \hat{g}) \beta_i = \lambda_i (\phi \circ \hat{g})$  for  $i = 1, 2, \dots, n$ . Since  $\|\hat{g} \circ \eta - \iota\|_{cb} < \varepsilon$  we conclude that

$$\|\alpha_i^* \phi_i|_V \beta_i - \lambda_i \phi|_V\| < 12\varepsilon$$

for  $i = 1, 2, \dots, n$ . Since this holds for any  $\varepsilon > 0$  and any finite-dimensional injective operator space  $V \subset Z$ , it follows by compactness and the fact that  $Z$  is a rigid rectangular  $\mathcal{OL}_{\infty,1+}$  space that the proper rectangular operator convex combination  $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is trivial. This concludes the proof that  $\phi$  is a rectangular operator extreme point.  $\square$

Fix now a separable nuclear operator space  $X$ , and consider the generic completely contractive map  $\Omega_{\text{NG}}^X : \text{NG} \rightarrow X$  as in Subsection 4.3. Then the characterization of such a map from Subsection 4.3 together with Proposition 8.2 shows that  $\Omega_{\text{NG}}^X$  is a complete facial quotient in the sense of Definition 8.1. Furthermore if  $X$  is a rigid rectangular  $\mathcal{OL}_{\infty,1+}$  space (and particularly when  $X = M_{n,m}(\mathbb{C})$  for some  $n, m \in \mathbb{C}$ ) one has that  $\Omega_{\text{NG}}^X$  is a rectangular operator extreme point. These observations together with the general results on universal morphisms from Section 4 and Section 5 conclude the proof of Theorem 1.4, Theorem 4.3, and Theorem 5.3.

**8.2. Exact operator systems.** An operator system can be defined as an operator space  $X$  with a distinguished element 1 (its *unit*) such that there exists a completely isometric linear map from  $X$  to the space  $B(H)$  of bounded linear operators on a Hilbert space that moreover maps the distinguished element 1 of  $X$  to the identity operator of  $H$ . Any operator system is endowed with an involution  $x \mapsto x^*$  coming from the inclusion  $X \subset B(H)$ . A linear map between operator systems is *unital* if it maps the unit to the unit. A unital linear completely contractive maps between operator systems is automatically *self-adjoint*, that is it commutes with taking adjoints.

An operator system can be regarded as a structure in the language of operator spaces with the addition of a constant symbol for the unit and a unary function symbol for the involution. The results from [14] show that operator systems form an axiomatizable class in this language. An earlier characterization of operator systems due to Choi and Effros involves the unit and the matrix positive cones [19]. In this setting morphisms will be unital completely contractive linear maps. Similarly embeddings will be unital completely isometric linear maps.

An operator system  $X$  is called *exact* if it is exact as an operator space or, equivalently, for every  $\delta > 0$  and finite-dimensional subspace  $E$  of  $X$ , there exists  $n \in \mathbb{N}$  and a *unital* completely contractive map  $f : X \rightarrow M_n$  such that  $\|id_K \otimes f^{-1}\| \leq 1 + \delta$ ; see [60, Section 5]. Any  $M_q$ -system  $X$  has a canonical (exact) operator system structure  $\text{OMIN}_q(X)$  obtained by setting

$$\|x\| = \sup_{\phi} \|(id_K \otimes \phi)(x)\|$$

for  $x \in \mathcal{K} \otimes X$ , where  $\phi$  ranges among all the unital  $q$ -contractive linear maps from  $X$  to  $M_q$ ; see [110]. The operator systems of the form  $\text{OMIN}_q(X)$  are called  $q$ -minimal in [110]. By the Arveson extension theorem [96, Theorem 7.5] the finite-dimensional injective operator systems are the finite  $\infty$ -sums of copies of  $M_n(\mathbb{C})$  for  $n \in \mathbb{N}$ . These are also precisely the finite-dimensional  $C^*$ -algebras.

Let now  $\mathcal{A}$  be the class of exact operator systems and, for every  $q \in \mathbb{N}$ ,  $\mathcal{A}_q$  be the class of  $q$ -minimal operator systems and  $\mathcal{I}_q$  the class of finite  $\infty$ -sums of copies of  $M_d(\mathbb{C})$  for  $d \leq q$  (these are precisely the finite-dimensional  $q$ -minimal injective operator systems). The class  $\mathcal{I}_\infty$  is the union of  $\mathcal{I}_q$  for  $q \in \mathbb{N}$ . Finally we let  $\mathcal{I}$  be the class of operator systems of the form  $M_n(\mathbb{C})$  for some  $n \in \mathbb{N}$ . One can verify as for operator spaces that the assumptions of Section 7 apply. The main difference lies in verifying Condition (2) of Subsection 2.3. As for  $M_q$ -systems, here one needs to approximate an approximately completely contractive self-adjoint unital linear map by a completely contractive unital linear map. This can be done using the following lemma. The *completely bounded norm*  $\|f\|_{cb}$  of a linear map between operator spaces  $f : X \rightarrow Y$  is the norm of  $\text{id}_{\mathcal{K}} \otimes f : \mathcal{K} \otimes X \rightarrow \mathcal{K} \otimes Y$ . Recall that a unital linear map between operator systems is completely contractive if and only if it is completely positive, i.e. for every  $n \in \mathbb{N}$  and positive element  $x$  of  $\mathcal{K} \otimes X$  the image  $(\text{id}_{\mathcal{K}} \otimes f)(x)$  is positive.

**Lemma 8.4.** *Suppose that  $V, W$  are operator systems, and  $f : V \rightarrow W$  is a self-adjoint linear map such that  $\|f\|_{cb} \leq 1 + \delta$ . If  $W$  is injective and  $f$  is unital, then there exists a unital completely positive linear map  $g : V \rightarrow W$  such that  $\|g - f\|_{cb} \leq 2\delta$ . If  $W$  is an arbitrary operator system,  $V$  has finite dimension  $n$ , and  $f$  is either unital or completely contractive, then there exists a unital completely positive linear map  $g : V \rightarrow W$  such that  $\|g - f\|_{cb} \leq 2n\delta$ .*

*Proof.* If  $W$  is injective, then we can assume without loss of generality that  $W = B(H)$ . In this case the first assertion follows from [15, Corollary B.9]. Suppose now that  $W \subset B(H)$  is an arbitrary operator system and  $f$  is unital. The proof in the case when  $f$  is completely contractive is analogous. By Wittstock's decomposition theorem [96, Theorem 8.5] there exist completely positive maps  $\phi_1, \phi_2 : V \rightarrow B(H)$  such that  $f = \phi_1 - \phi_2$  and  $\|\phi_1 + \phi_2\|_{cb} \leq \|f\|_{cb} \leq 1 + \delta$ . In particular by [96, Proposition 3.2] we have that

$$\|\phi_1(1)\| \leq \|\phi_1(1) + \phi_2(1)\| \leq \|\phi_1 + \phi_2\|_{cb} \leq 1 + \delta.$$

Since  $\phi_1(1) - \phi_2(1)$  is the identity operator on  $H$ , this implies that  $\|\phi_2\|_{cb} = \|\phi_2(1)\| \leq \delta$ .

By [36, Lemma 2.4] there exists a positive linear functional  $\theta$  on  $V$ , which can we regard as a function  $\theta : V \rightarrow W$ , such that  $\theta - \phi_2$  is completely positive and  $\|\theta\| \leq n\delta$ . Consider now the completely positive map  $g_0 = f + \theta = \phi_1 + (\theta - \phi_2)$  and observe that  $\|g_0 - f\| \leq n\delta$ . Set  $g(x) := g_0(x) + \tau(x)(g_0(1) - 1)$ , where  $\tau$  is a state on  $V_0$ . Then  $g$  is a unital completely positive map such that  $\|g - f\|_{cb} \leq 2n\delta$ .  $\square$

Lemma 8.4 shows that Condition (2) of Subsection 2.3 holds for operator systems with  $\varpi(\delta) = 2\delta$ . As basic tuples one can consider in this context linearly independent tuples whose first element is the unit. To verify that the assumptions of Subsection 2.2 are satisfied one can use Lemma 8.4 together with the small perturbation argument [98, Lemma 2.13.2]. An operator system is approximately injective according to Definition 3.2 if and only if it is nuclear. A (rigid)  $\mathcal{I}_\infty$ -structure as in Definition 3.6 is an operator system which is a (rigid)  $\mathcal{OL}_{\infty,1+}$  space in the sense of [56]. This follows from Lemma 8.4 together with the following lemma, which can be proved as [32, Lemma 2.6].

**Lemma 8.5.** *Suppose that  $X, Y$  are operator systems and  $\phi : X \rightarrow Y$  a completely positive map such that  $\|\phi\|_{cb} \leq 1 + \delta < 2$ . Consider a state  $\tau$  of  $X$ . If  $\psi : X \rightarrow Y$  is defined by  $\psi(x) = \phi(x) + \tau(x)(1 - \phi(1))$ , then  $\psi$  is an injective unital completely positive map such that  $\|\psi^{-1}\| \leq (1 + \delta)(1 - \delta)^{-1}$ .*

A separable operator system is a rigid  $\mathcal{OL}_{\infty,1+}$  space if and only if it is unital isometrically isomorphic to the limit of an inductive sequence of finite-dimensional  $C^*$ -algebras with unital completely isometric connective maps. This is a consequence of the following lemma, which can be proved similarly as [62, Lemma 7.1] using [12, Proposition 4.2.8].

**Lemma 8.6.** *Suppose that  $B$  is a finite-dimensional  $C^*$ -algebra and  $\varepsilon > 0$ . Then there exists  $\delta = \delta_p(\varepsilon, B)$  such that for any finite-dimensional  $C^*$ -algebra  $A$  and injective linear map  $\phi : B \rightarrow A$  such that  $\|\phi\| \leq 1 + \delta$ ,  $\|\phi^{-1}\| \leq 1 + \delta$ , and  $\|\phi(1) - 1\| \leq \delta$ , there exists a complete order embedding  $\psi : B \rightarrow A$  such that  $\|\psi - \phi\|_{cb} \leq \varepsilon$ .*

It follows from the discussion above that finite-dimensional operator systems form a Fraïssé class. We will call the corresponding limit  $A(\mathbb{NG})$  the *noncommutative Poulsen system*. The matrix state space  $\mathbb{NG}$  of the operator system  $A(\mathbb{NG})$  will be called the *noncommutative Poulsen simplex*. Since  $A(\mathbb{NP})$  is a separable nuclear operator system—and, in fact, a rigid  $\mathcal{OL}_{\infty,1+}$ -space— $\mathbb{NP}$  is a metrizable noncommutative Choquet simplex in the sense of [28]. The noncommutative Poulsen simplex satisfies the natural noncommutative analog of the

defining property of the Poulsen simplex: the set of matrix extreme points of  $\mathbb{NP}$  is dense in  $\mathbb{NP}$ . It is furthermore proved in [28] that  $\mathbb{NP}$  is the unique metrizable noncommutative Choquet simplex with such a property.

The operator system  $A(\mathbb{NP})$  associated with the noncommutative Poulsen simplex is the first example of a separable exact—in fact, nuclear—operator system that contains a unital completely isometric copy of any other separable exact operator system. It is furthermore proved in [28] that  $A(\mathbb{NP})$  is the unique separable nuclear operator system that is universal in the sense of Kirchberg and Wassermann [63]. The model-theoretic properties of the noncommutative Poulsen system  $A(\mathbb{NP})$  have been considered in [49], where it is shown that  $A(\mathbb{NP})$  is the unique separable existentially closed operator system, and the unique prime model of its first order theory. Furthermore, an operator system is nuclear if and only if it is positively existentially closed.

In analogy with the case of function systems, we consider the following notion of face for compact matrix convex sets.

**Definition 8.7.** *A unital complete facial quotient mapping  $P : Z \rightarrow X$  between operator systems is a unital complete quotient mapping whose kernel is a complete  $M$ -ideal.*

Suppose that  $\mathbf{K}$  is a compact matrix convex set. The notion of closed matrix face of  $\mathbf{K}$  can be defined in terms of unital facial quotients. By definition, a compact matrix convex subset  $\mathbf{F}$  of  $\mathbf{K}$  is a closed matrix face if the induced map  $A(\mathbf{K}) \rightarrow A(\mathbf{F})$  is a unital complete facial quotient in the sense of Definition 8.7.

We say that an operator system  $A$  satisfies the operator metric approximation property if it satisfies such a property as an operator space. It follows from Lemma 8.6 that this is equivalent to the assertion that the identity map of  $A$  is the pointwise limit of finite rank unital completely positive maps. The similar proof as Proposition 8.2 gives the following result.

**Proposition 8.8.** *Suppose that  $X, Y$  are operator systems, and  $P : Z \rightarrow X$  is a unital complete quotient mapping. The following statements are equivalent:*

- (1)  *$P$  is a unital complete facial quotient;*
- (2) *whenever  $\varepsilon > 0$ ,  $E \subset F$  are finite-dimensional operator systems,  $g : F \rightarrow X$  is a unital completely positive map, and  $f : E \rightarrow Z$  is a unital complete isometry such that  $P \circ f = g|_E$ , then there exists a unital completely positive map  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} \leq \varepsilon$ ;*
- (3) *whenever  $\varepsilon > 0$ ,  $A$  is a separable operator systems with the operator metric approximation property,  $E \subset A$  is a finite-dimensional subsystem, and  $f : E \rightarrow Z$  and  $g : A \rightarrow X$  are unital completely positive maps such that  $\|P \circ f - g|_E\|_{cb} < \varepsilon$ , then there exists a unital completely positive map  $\hat{g} : A \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} < 3\varepsilon$ .*

*If furthermore  $Z$  is exact and  $X$  is nuclear, then these are also equivalent to:*

- (4) *for any  $\varepsilon > 0$ ,  $q \in \mathbb{N}$ , finite-dimensional  $q$ -minimal operator systems  $E \subset F$ , and unital completely positive maps  $f : E \rightarrow Z$  and  $g : F \rightarrow X$  such that  $P \circ f = g|_E$ , there exists a unital completely positive map  $\hat{g} : F \rightarrow Z$  such that  $P \circ \hat{g} = g$  and  $\|\hat{g}|_E - f\|_{cb} < \varepsilon$ .*

The implication (1) $\Rightarrow$ (3) of Proposition 8.8 can be proved similarly as [38, Theorem 5.2], where one starts from [18, Proposition 2.2] instead of [38, Lemma 5.1]. For the implication (4) $\Rightarrow$ (1) one can use Lemma 8.4.

Suppose that  $X \subset B(H)$  and  $Z$  are operator systems. An *operator convex combination* as defined in [44] is an expression  $\alpha_1^* \phi_1 \alpha_1 + \cdots + \alpha_n^* \phi_n \alpha_n$  where  $\phi_i : Z \rightarrow B(H_i)$  are unital completely positive maps, and  $\alpha_i : K \rightarrow K_i$  are linear maps of norm at most 1. We say that  $\alpha_1^* \phi_1 \alpha_1 + \cdots + \alpha_n^* \phi_n \alpha_n$  is a *proper* operator convex combination if  $\alpha_i$  are surjective and  $\alpha_1^* \alpha_1 + \cdots + \alpha_n^* \alpha_n = 1$ . It is clear that when  $H$  is finite-dimensional the notion of proper operator convex combination coincides with the notion of matrix convex combination considered in [106, 42]. A proper rectangular operator convex combination  $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is *trivial* if  $\alpha_i^* \alpha_i = \lambda_i 1$ ,  $\beta_i^* \beta_i = \lambda_i 1$ , and  $\alpha_i^* \phi_i \beta_i = \lambda_i \phi$  for some  $\lambda_i \in [0, 1]$ . Then a unital completely positive map  $\phi : Z \rightarrow X$  is an *operator extreme point* if any proper operator convex combination  $\phi = \alpha_1^* \phi_1 \beta_1 + \cdots + \alpha_n^* \phi_n \beta_n$  is trivial. Theorem A of [42] shows that when  $H$  is finite-dimensional, an operator extreme point is the same as a matrix extreme point as defined in [106, 42]. We observe that, if  $A$  is a unital  $C^*$ -algebra, then the identity map  $A \rightarrow A$  is an operator extreme point by [4, Corollary 1.4.3].

The same proof as Proposition 8.3 gives the following result.

**Proposition 8.9.** *Suppose that  $Z$  and  $X$  are rigid  $\mathcal{OL}_{\infty, 1+}$  systems, and  $\phi : Z \rightarrow X$  is a unital complete facial quotient. Then  $\phi$  is an operator extreme point.*

One can now deduce Theorem 1.4 (1)–(3), Theorem 4.4, and Theorem 5.5 from Proposition 8.8, Proposition 8.9 and the general results from Section 7, Section 4, and Section 5. Indeed, suppose that  $\mathbf{F}$  is a metrizable noncommutative Choquet simplex. Recall that this means that  $\mathbf{F}$  is the matrix state space of a separable

nuclear operator system  $A(\mathbf{F})$ . Consider the generic completely positive map  $\Omega_{A(\mathbb{N}\mathbb{G})}^{A(\mathbf{F})} : A(\mathbb{N}\mathbb{G}) \rightarrow A(\mathbf{F})$  as constructed in Section 5. The characterization of  $\Omega_{A(\mathbb{N}\mathbb{G})}^{A(\mathbf{F})}$  from Section 5 together with the equivalence of (1) and (4) in Proposition 8.8 show that  $\Omega_{A(\mathbb{N}\mathbb{G})}^{A(\mathbf{F})}$  is a unital complete facial quotient mapping, and hence the dual map induces an inclusion of  $\mathbf{F}$  inside the noncommutative Poulsen simplex as a noncommutative face. The other assertions are proved analogously.

It remains to prove that the canonical action of the group  $\text{Aut}(\mathbb{N}\mathbb{P})$  of matrix affine homeomorphisms of  $\mathbb{N}\mathbb{P}$ —which can be identified with the space of surjective unital complete isometries of  $A(\mathbb{N}\mathbb{P})$  endowed with the topology of pointwise convergence—on the space  $S_1(A(\mathbb{N}\mathbb{P}))$  of states of the noncommutative Poulsen system is minimal. In view of Corollary 5.9, this is a consequence of the following lemma.

**Lemma 8.10.** *Fix  $d \in \mathbb{N}$  and  $\varepsilon > 0$ . There exists  $m \in \mathbb{N}$  such that for any  $s \in S_1(M_d(\mathbb{C}))$  and  $t \in S_1(M_m(\mathbb{C}))$  there exists a complete order embedding  $\phi : M_d(\mathbb{C}) \rightarrow M_m(\mathbb{C})$  such that  $\|s \circ \phi - t\| \leq \varepsilon$ .*

*Proof.* Pick  $\ell \in \mathbb{N}$  such that  $1/\ell \leq \varepsilon/16$ . Let  $\mathcal{P}$  be a finite set of positive elements of  $M_d(\mathbb{C})$  of norm at most 1 with the property that for any positive element  $x$  of  $M_d(\mathbb{C})$  of norm at most 1 there exists  $x_0 \in \mathcal{P}$  such that  $\|x - x_0\| < \eta$ . Consider  $k \in \mathbb{N}$  such that  $k > \ell |\mathcal{P}|$  and set  $m := kd$ . Suppose that  $s \in S_1(M_{kd}(\mathbb{C}))$  and  $t \in S_1(M_d(\mathbb{C}))$ . Then there exists a positive matrix  $a \in M_{kd}(\mathbb{C})$  such that  $\text{Tr}_{kd}(a) = 1$  and  $s(x) = \text{Tr}_{kd}(ax)$  for every  $x \in M_{kd}(\mathbb{C})$ , where  $\text{Tr}_{kd}$  denotes the usual trace on  $M_{kd}(\mathbb{C})$ . We regard  $M_{kd}(\mathbb{C})$  as the space of bounded linear operators on the space  $\mathbb{C}^{kd}$  with canonical basis  $(e_1, \dots, e_{kd})$ . For  $1 \leq i \leq k$ , let  $p_i$  be the orthogonal projection on the span of  $\{e_{(i-1)d+1}, \dots, e_{id}\}$ , and  $a_i = p_i a p_i$ . Suppose that  $b$  is an element of  $\mathcal{P}$ , and define  $b = p_1 b p_1 + \dots + p_k b p_k$ . Then

$$s(B) = \text{Tr}_m(ab) = \sum_{i=1}^{\ell} \text{Tr}_d(a_i b) \leq 1,$$

where  $\text{Tr}_d$  denotes the canonical trace on  $M_d(\mathbb{C})$ . Therefore the set of  $i \in \{1, 2, \dots, k\}$  such that  $\text{Tr}_d(a_i b) \geq 1/\ell$  contains at most  $\ell$  elements. Therefore the set of  $i \in \{1, 2, \dots, k\}$  such that  $\text{Tr}_d(a_i b) \geq 1/\ell$  for every  $b \in \mathcal{P}$  contains at most  $\ell |\mathcal{P}|$  elements. Therefore there exists  $i \in \{1, 2, \dots, k\}$  such that  $\text{Tr}_d(a_i b) \leq 1/\ell$  for every  $b \in \mathcal{P}$ . Without loss of generality we can assume that  $i = 1$ . We also have  $\text{Tr}_d(a_1 b) \leq 8/\ell$  for every  $b \in M_d(\mathbb{C})$  of norm at most 1, since  $\mathcal{P}$  is  $1/\ell$ -dense in the set of positive elements of  $M_d(\mathbb{C})$  of norm at most 1. Therefore  $\|a_1\| \leq 8/\ell$  and  $\left| \sum_{i=2}^{\ell} \text{Tr}_d(a_i) - 1 \right| \leq 8/\ell$ . Define now the complete order embedding  $\phi : M_d(\mathbb{C}) \rightarrow M_{kd}(\mathbb{C})$  by

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & t(x)I_{(k-1)d} \end{bmatrix}$$

where  $I_{(k-1)d}$  is the identity  $(k-1)d \times (k-1)d$  matrix. Observe that, for every  $x \in M_d(\mathbb{C})$  of norm at most 1,

$$|s(\phi(x)) - t(x)| = \left| \text{Tr}(ax) + t(x) \sum_{i=2}^{\ell} \text{Tr}_d(a_i) - t(x) \right| \leq |\text{Tr}(ax)| + \left| \sum_{i=2}^{\ell} \text{Tr}_d(a_i) - 1 \right| \leq 16/\ell.$$

This concludes the proof.  $\square$

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